Unfolding 3D stories from 2D mathematical diagrams using dynamic geometry

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RSME-SCM-SEMA-SIMAI-UMI

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T: Let’s try to extract a vertex from the plane.
Imagine to redefine... rather, let’s make two things.
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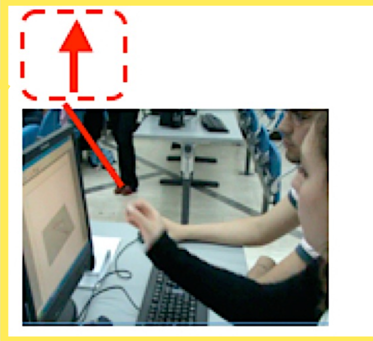

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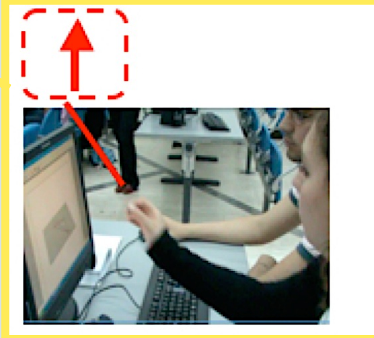

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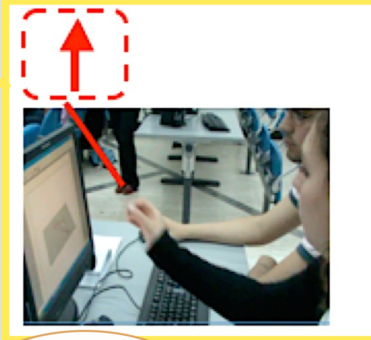
3D



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3D

DGS

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3D

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WHICH STORY ARE WE TELLING?

What do you see looking at/in the water?

A photo from the seaside around the West End

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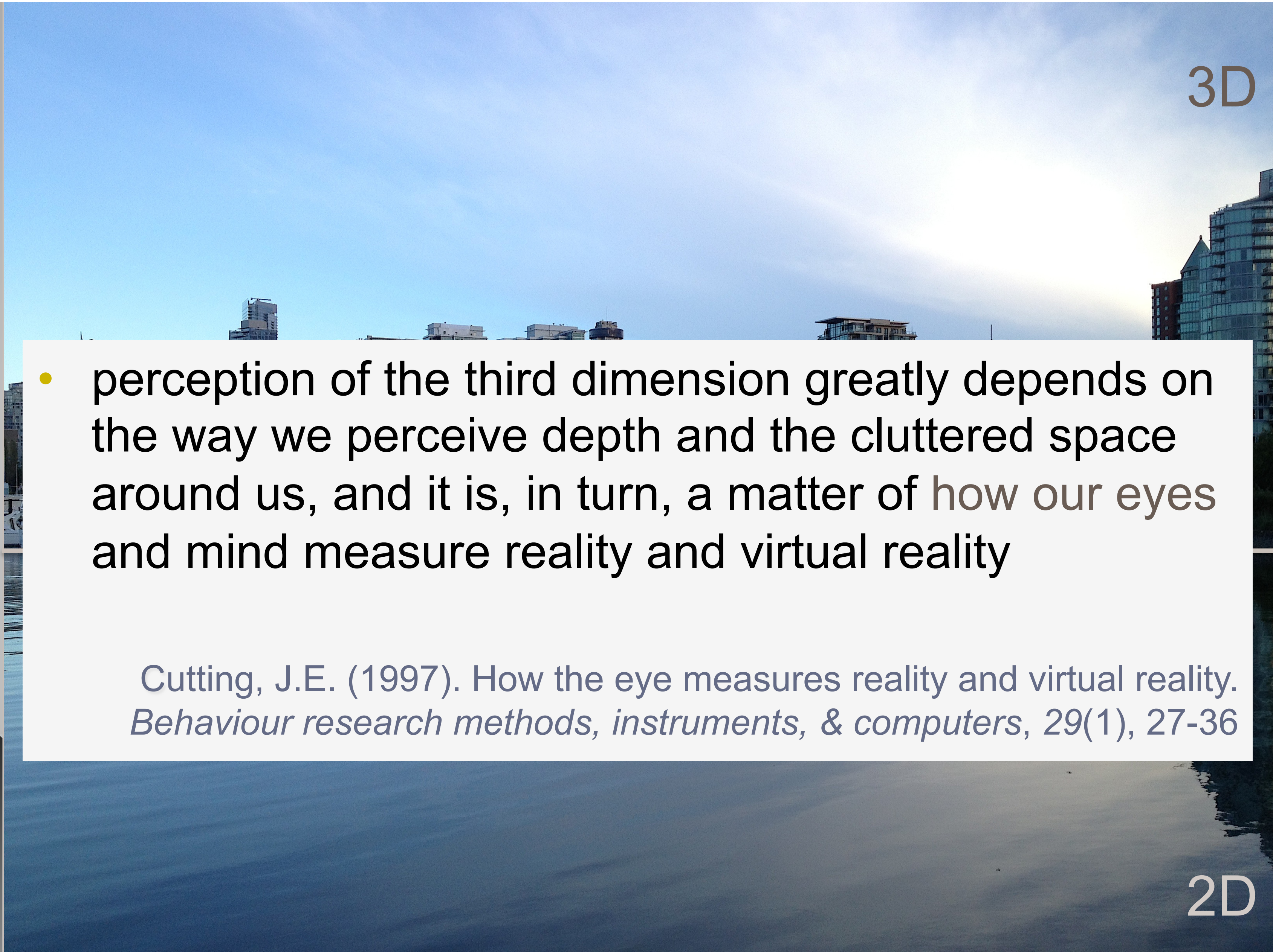
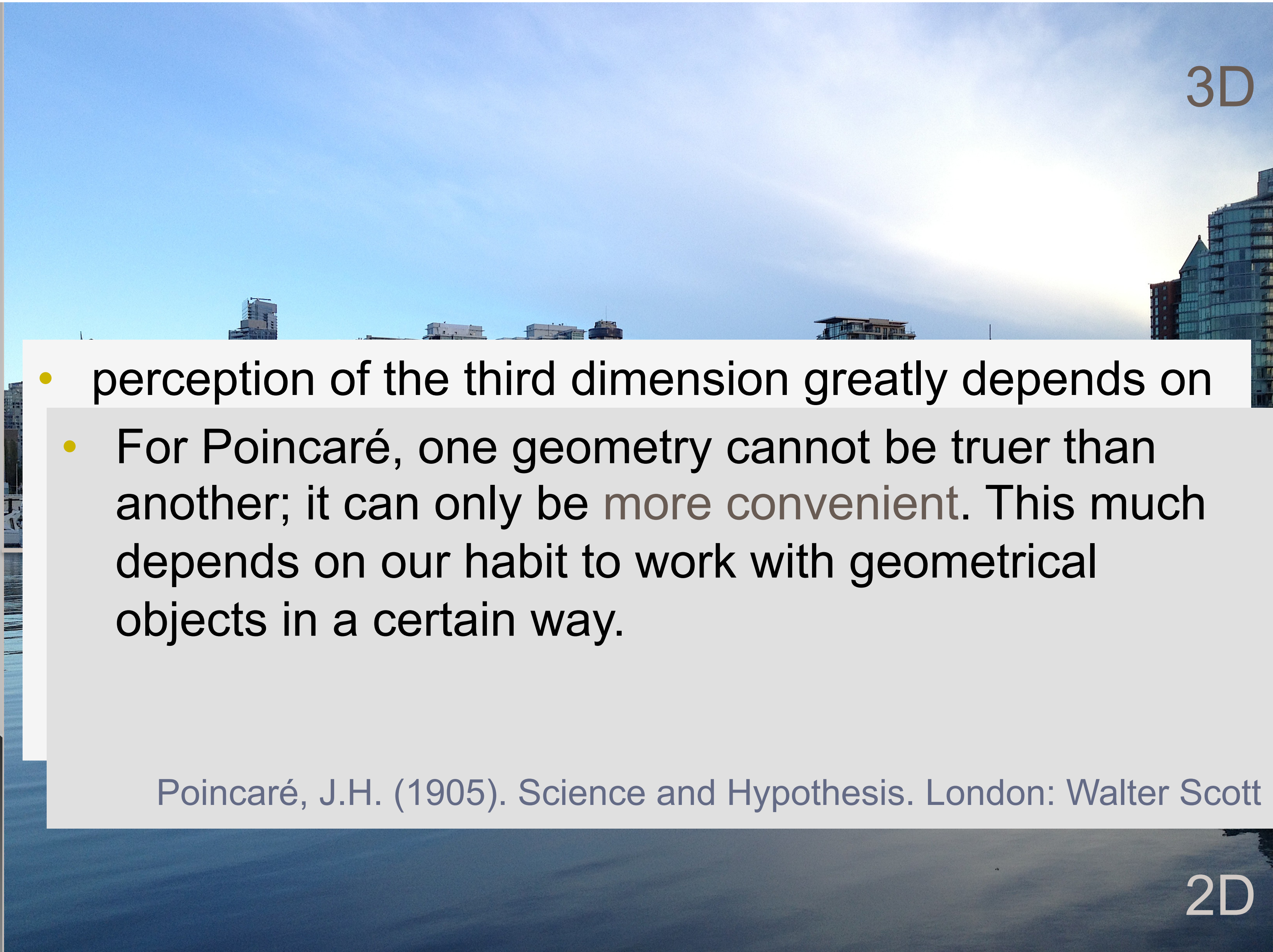


- perception of the third dimension greatly depends on the way we perceive depth and the cluttered space around us, and it is, in turn, a matter of how our eyes and mind measure reality and virtual reality

Cutting, J.E. (1997). How the eye measures reality and virtual reality. *Behaviour research methods, instruments, & computers*, *29*(1), 27-36

A photo from the seaside around the West End



- perception of the third dimension greatly depends on
- For Poincaré, one geometry cannot be truer than another; it can only be more convenient. This much depends on our habit to work with geometrical objects in a certain way.

Poincaré, J.H. (1905). Science and Hypothesis. London: Walter Scott

A photo from the seaside around the West End

"I am a research mathematician, working in discrete applied geometry. My own practice of mathematics is deeply visual: the problems I pose; the methods I use; the ways I find solutions; the way I communicate my results. The visual is central to mathematics as I experience it. It is not central to mathematics as many teachers present it nor as students witness it. This contrast is striking."

Whiteley, W. (2004). Visualization in mathematics: Claims and questions towards a research program. *Paper presented at the 10th International Congress on Mathematical Education*. Copenhagen, Denmark: July 4-11, 2004

In mathematics

Visual thinking is essential for professional mathematicians.

Healey, L. & Hoyles, C. (1999). Visual and Symbolic Reasoning in Mathematics: Making Connections with Computers?. *Mathematical Thinking and Learning, 1*(1), 59-84

Mathematics as a visible enterprise (?)

… the emergence of *"effective pedagogy that can enhance the use and power of visualization in mathematics education."*

- pedagogical in(ter)vention

Presmeg, N. (2006). Research on visualization in learning and teaching mathematics. In A. Gutierrez & P. Boero (Eds.), *Handbook of Research on the Psychology of Mathematics Education. Past, Present and Future* (pp. 205-235). Rotterdam/ Tapei: Sense Publisher

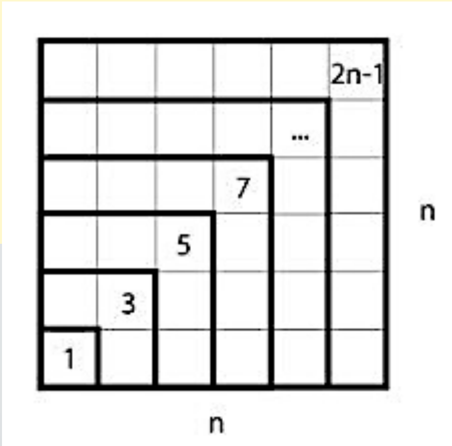
Visualization in communication

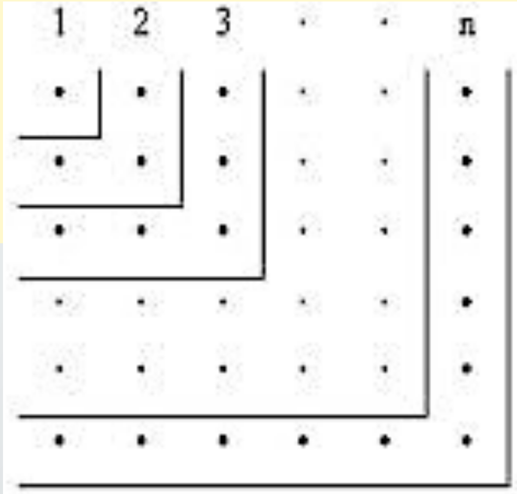
"in spite of the famous "intangibility" of mathematical objects, mathematical communication depends on what we see no less than do other, less abstract types of talk."

Sfard, A. (2008). *Thinking as communicating: Human development, the growth of discourse, and mathematizing*. Cambridge, MA: Cambridge University Press

Example 1

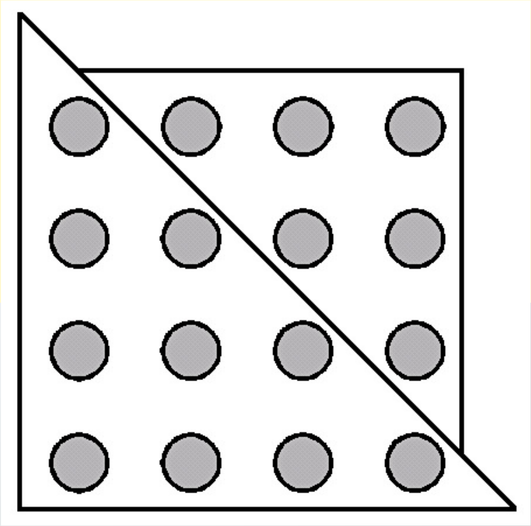
A square number is immediately seen as the sum of odd numbers through suitable diagrams





Example 2

A square number is immediately seen as the sum of two triangular numbers



Spatial Geometry ?!?

The general reputation that spatial geometry is difficult is usually connected to the feeling that *seeing* in 3D is difficult.

Bakó, M. (2003). Different projecting methods in teaching spatial geometry. In M.A. Mariotti (Ed.), Proceedings of the 3rd Congress of the European Society for Research in Mathematics Education. Bellaria, Italy: February 28-March 3, 2003

Spatial Geometry ?!?

The general reputation that spatial geometry is difficult is usually connected to the feeling that *seeing* in 3D is difficult.

- poor confidence of teachers

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Spatial Geometry ?!?

The general reputation that spatial geometry is difficult is usually connected to the feeling that *seeing* in 3D is difficult.

- not convenient, neglected, avoided

Bakó, M. (2003). Different projecting methods in teaching spatial geometry. In M.A. Mariotti (Ed.), Proceedings of the 3rd Congress of the European Society for Research in Mathematics Education. Bellaria, Italy: February 28-March 3, 2003

Seeing ???

Ontological difference: three-dimensional figures vs. bi-dimensional diagrams that embody them

Our research

Seeing ???

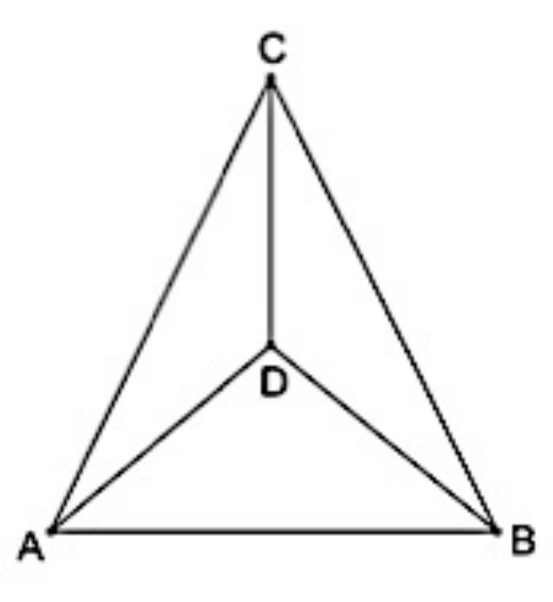
Ontological difference: three-dimensional figures vs. bi-dimensional diagrams that embody them

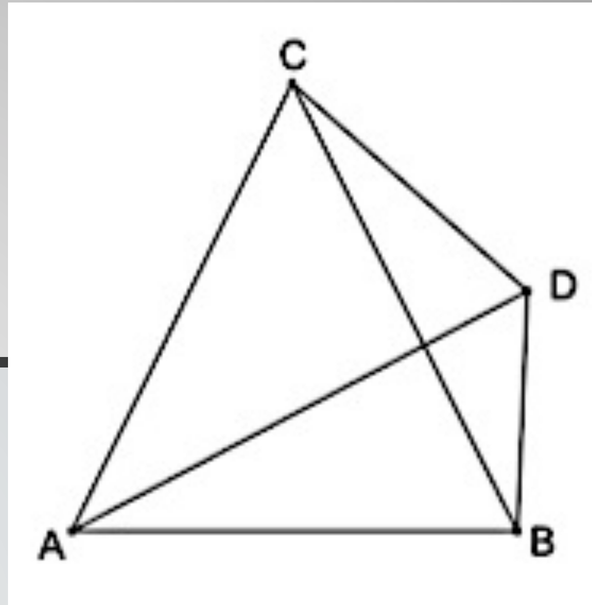
- focus on the visual challenges involved in the study of objects in space and on the role of technology to address such challenges

Our research

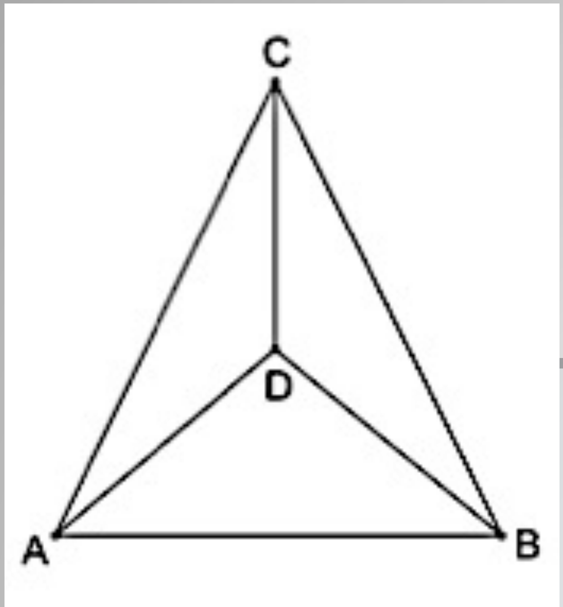
Various aspects

- one geometry cannot be truer than another; it can only be more convenient (Poincaré)
- conflicts between knowing and seeing (Parzysz)
- conflicts due to the perception of the third dimension in flat diagrams (Oldknown)
- figural vs. conceptual aspects (Fishbein)

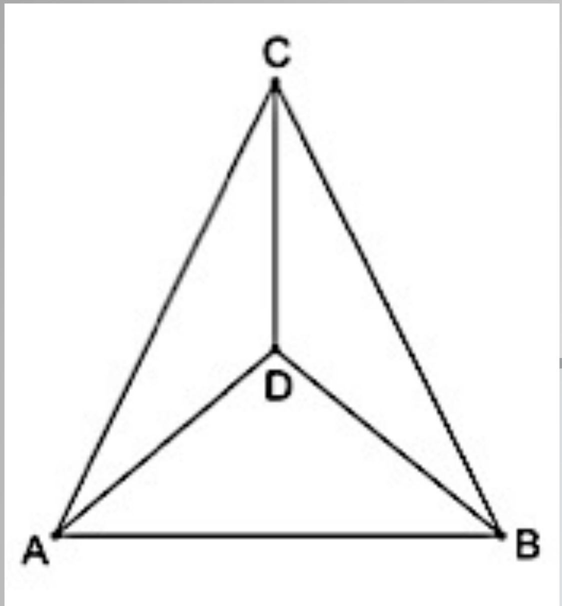




What do you see looking at the pictures?



Looking at the picture on the left?

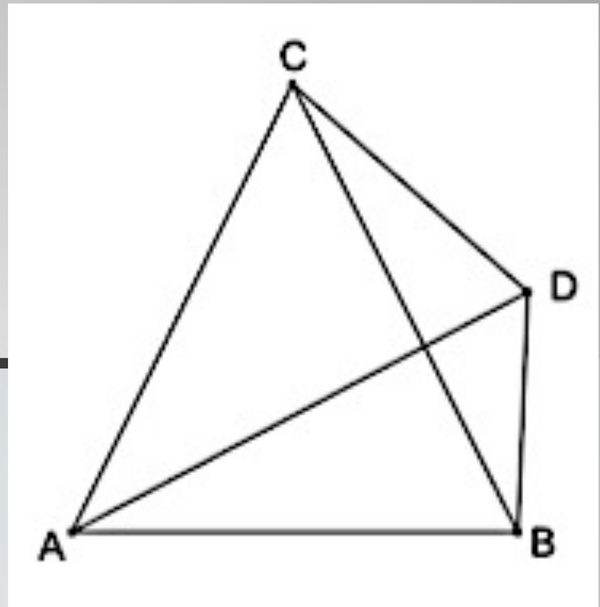


Looking at the picture on the left?

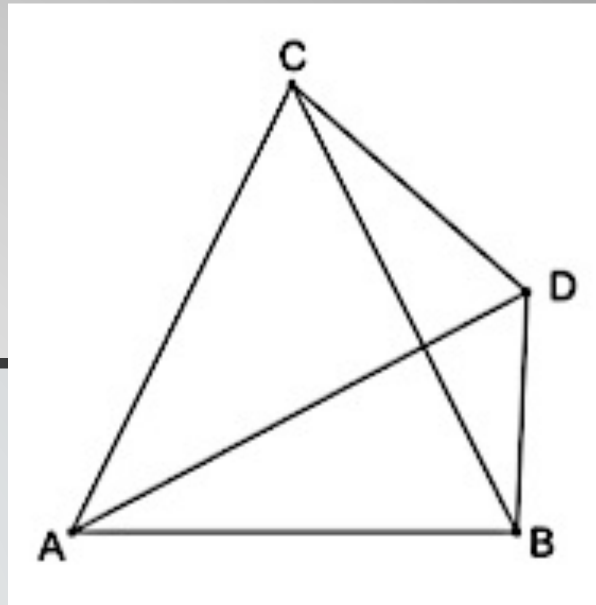
… a tetrahedron, based on the triangle ABC

… a tetrahedron, based on the triangle ABD

… a concave quadrilateral ADBC, with diagonals AB, DC



And looking at the picture on the right?



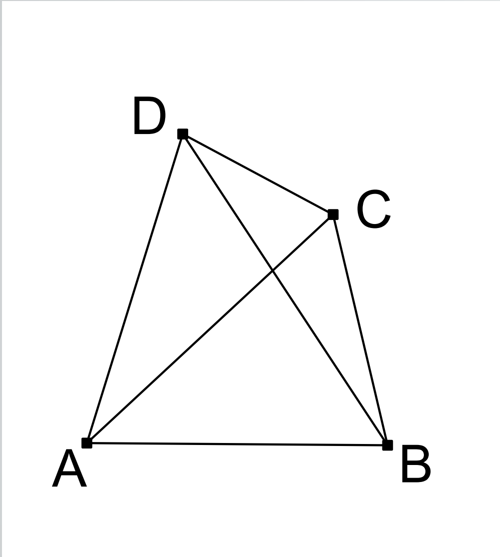
And looking at the picture on the right?

… a tetrahedron, based on the triangle ABD

… a convex quadrilateral ABDC, with diagonals AD and BC

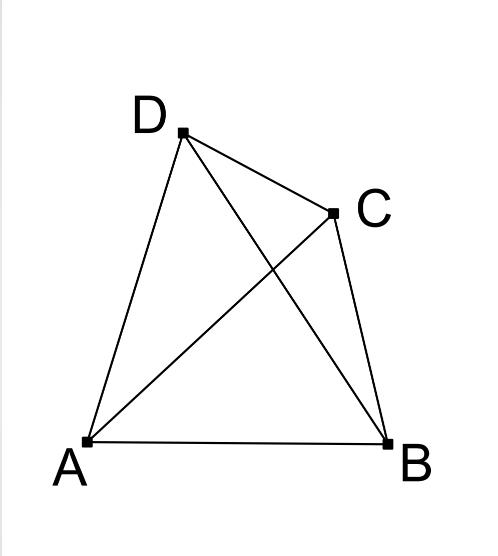
… nothing else than four points and six segments!

“Analogy” between two figures

Quadrilaterals	Tetrahedra
	

“Analogy” between two figures

Quadrilaterals Tetrahedra



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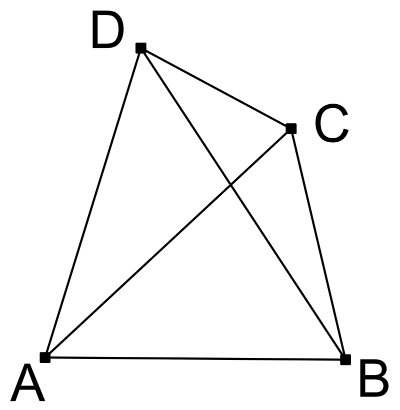
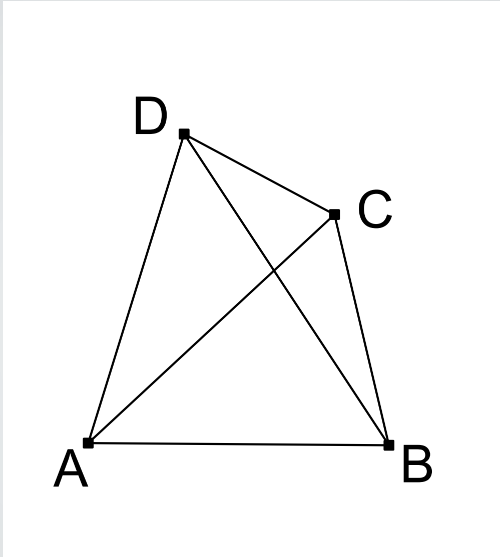


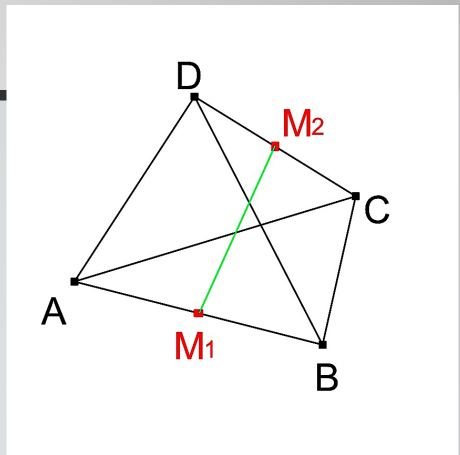
“Analogy” between two figures

Quadrilaterals	Tetrahedra
four *vertices*	four *vertices*
six *edges*	six *edges*
four *faces*	four *faces*

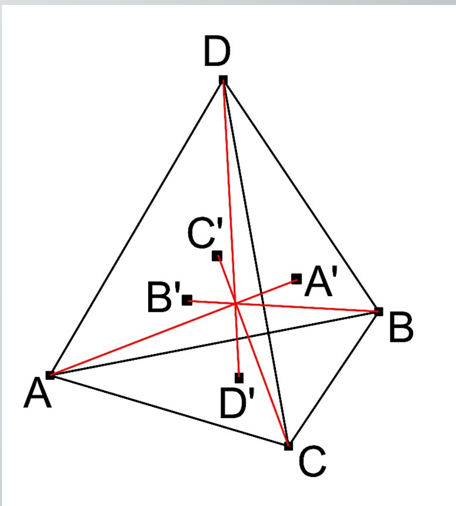


Using the "analogy"

Bimedian of *F:* segment that joins the midpoints of two opposite edges



Median of *F:* segment that joins one vertex with the centroid of the opposite face



Using the “analogy”

Property A.

1. The three bimedians of *F* all pass through one point (*centroid*).
2. The centroid of *F* bisects each bimedian.

Property B.

1. The four medians of *F* meet in its centroid.
2. The centroid of *F* divides each median in the ratio 1:3, the longer segment being on the side of the vertex of *F*.

Back to the start

May 2012, group of 12 university students (age 22)

Foundations of Mathematics class

group work (two/three students)

use of CABRI II Plus and CABRI 3D

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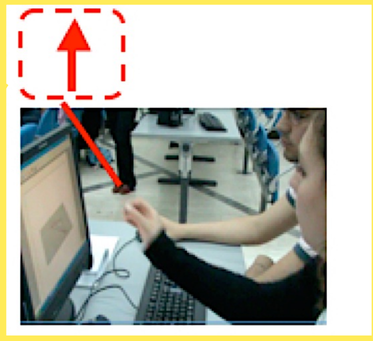

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Dragging vs. Glassball

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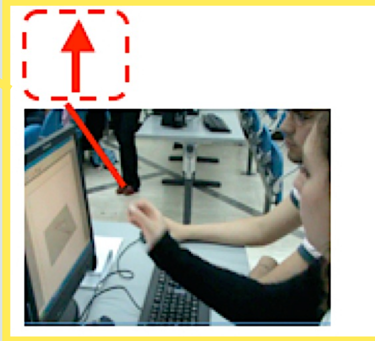

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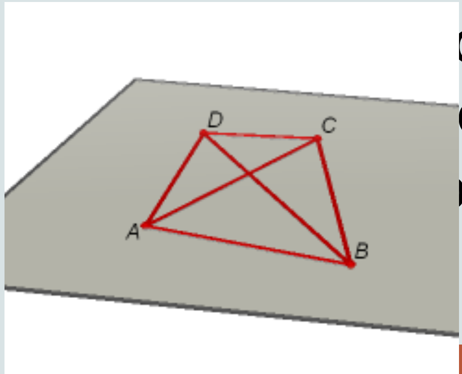

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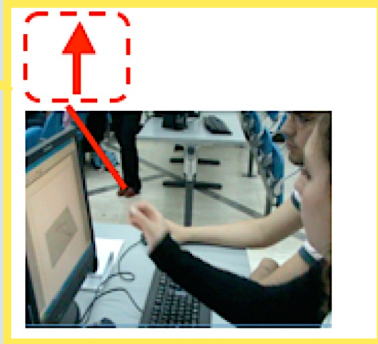

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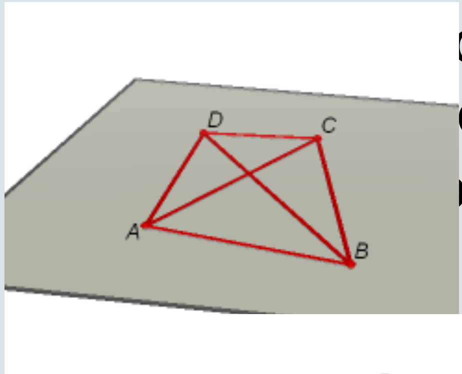

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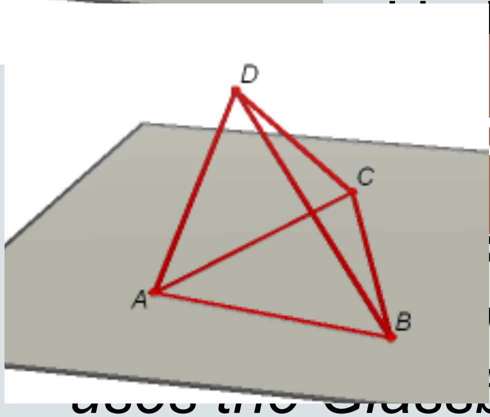
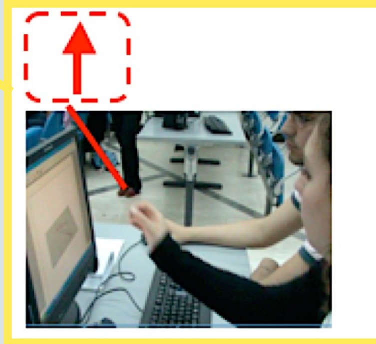


Glassball



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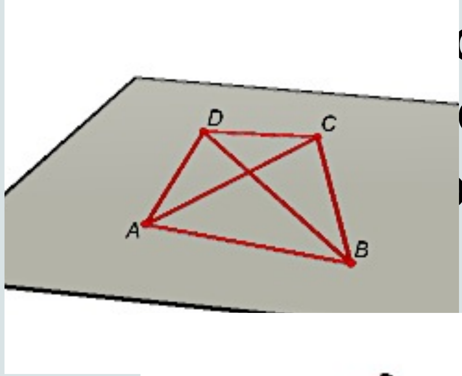

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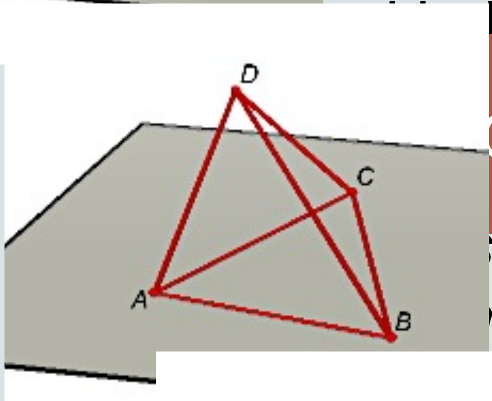

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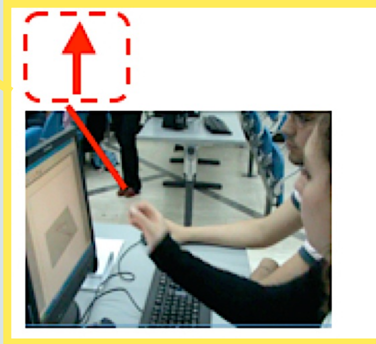

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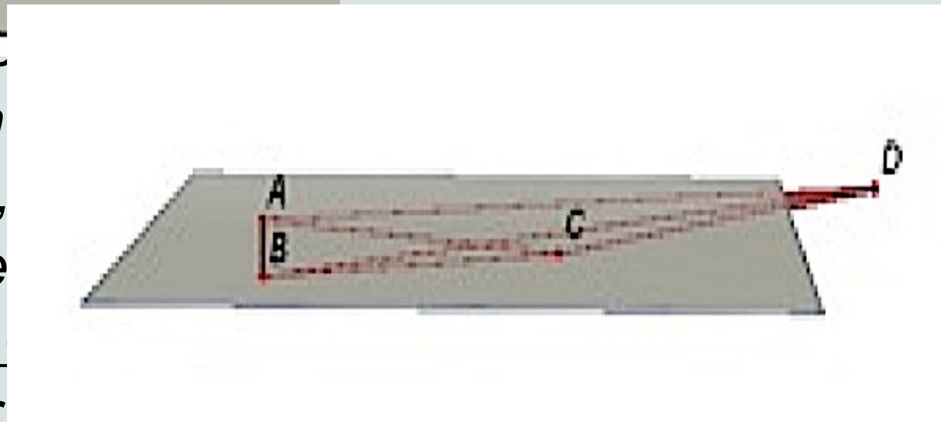

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Story 1

Knowing vs. seeing

Dragging vs. Glassball

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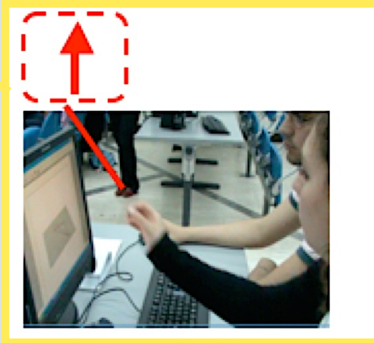

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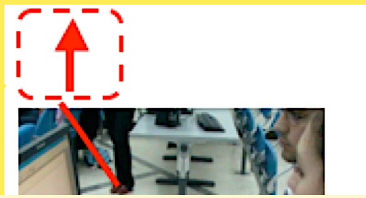

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base. <*M drags a vertex to see the*

The problems of coding a 3D geometrical figure into a single drawing have their origin in the impossibility of giving a close representation of it, and in the subsequent obligation of 'falling back' on a distant representation […] an insoluble dilemma, due to the fact that what one ***knows*** *of a 3D object comes into conflict with what one* ***sees*** *of it.*

(Parzysz, 1988)

T: What is the centre?

V: The point where the diagonals meet.

Story 1

T: Let's try to extract a vertex from the plane.
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Knowing …

Redefinition tool

V: It shape… …angular base.

Dragging vs. Glass…

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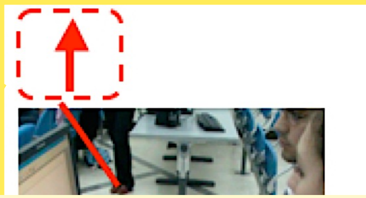

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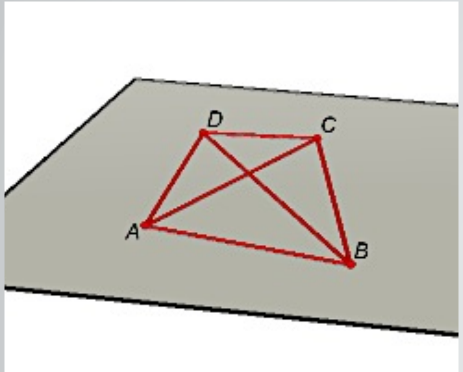
Story 1

V: The point where the diagonals meet.

Redefinition tool



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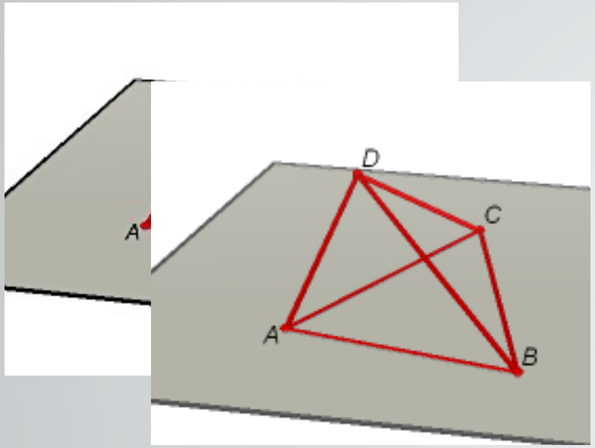

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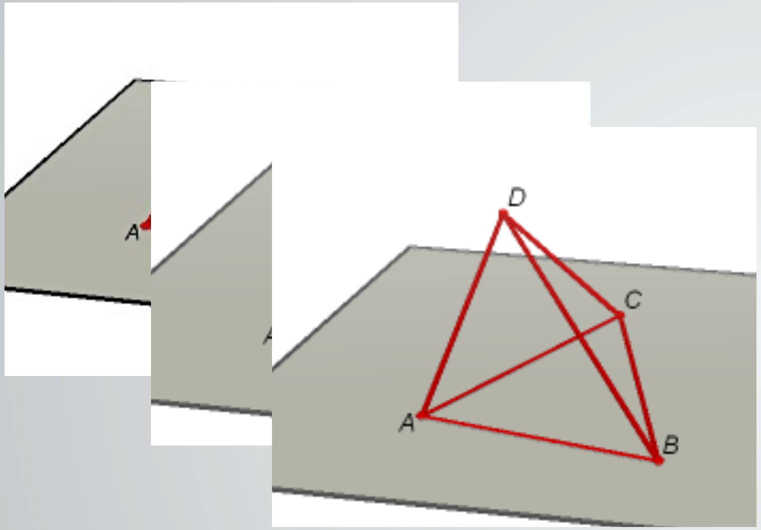

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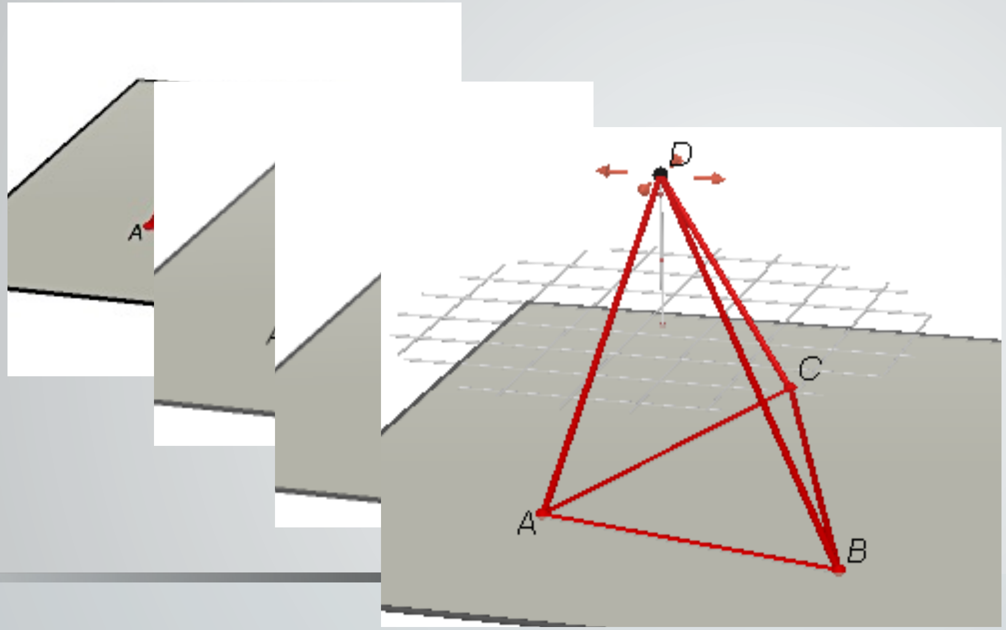

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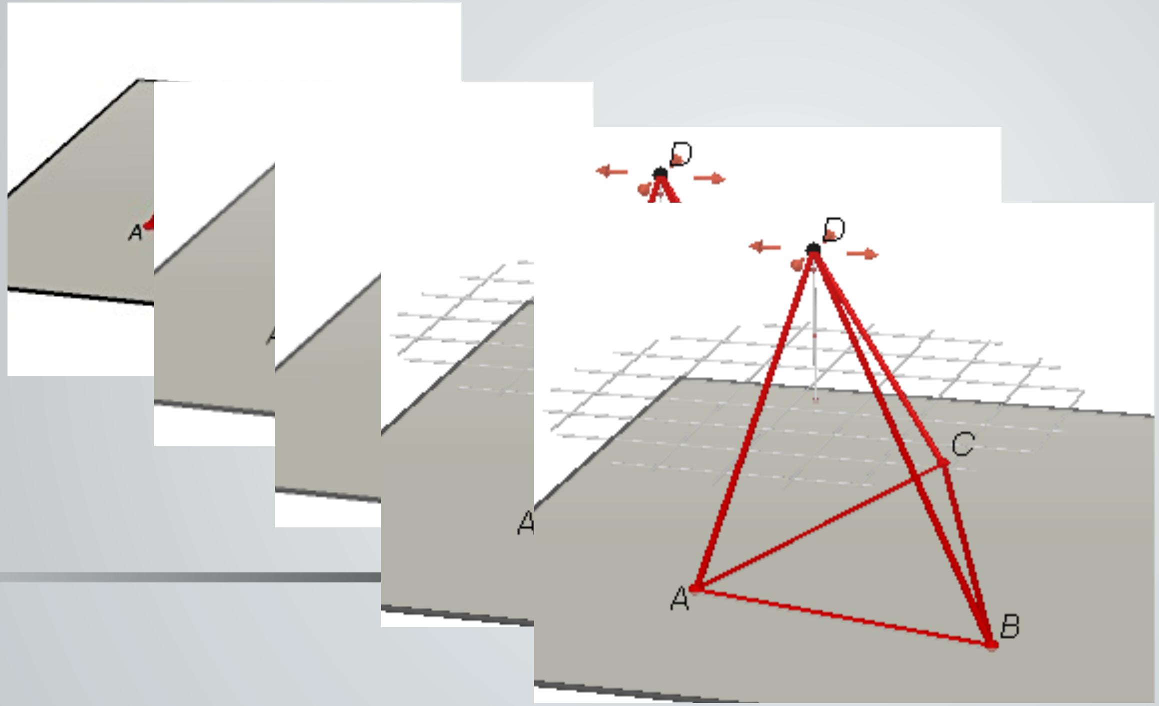

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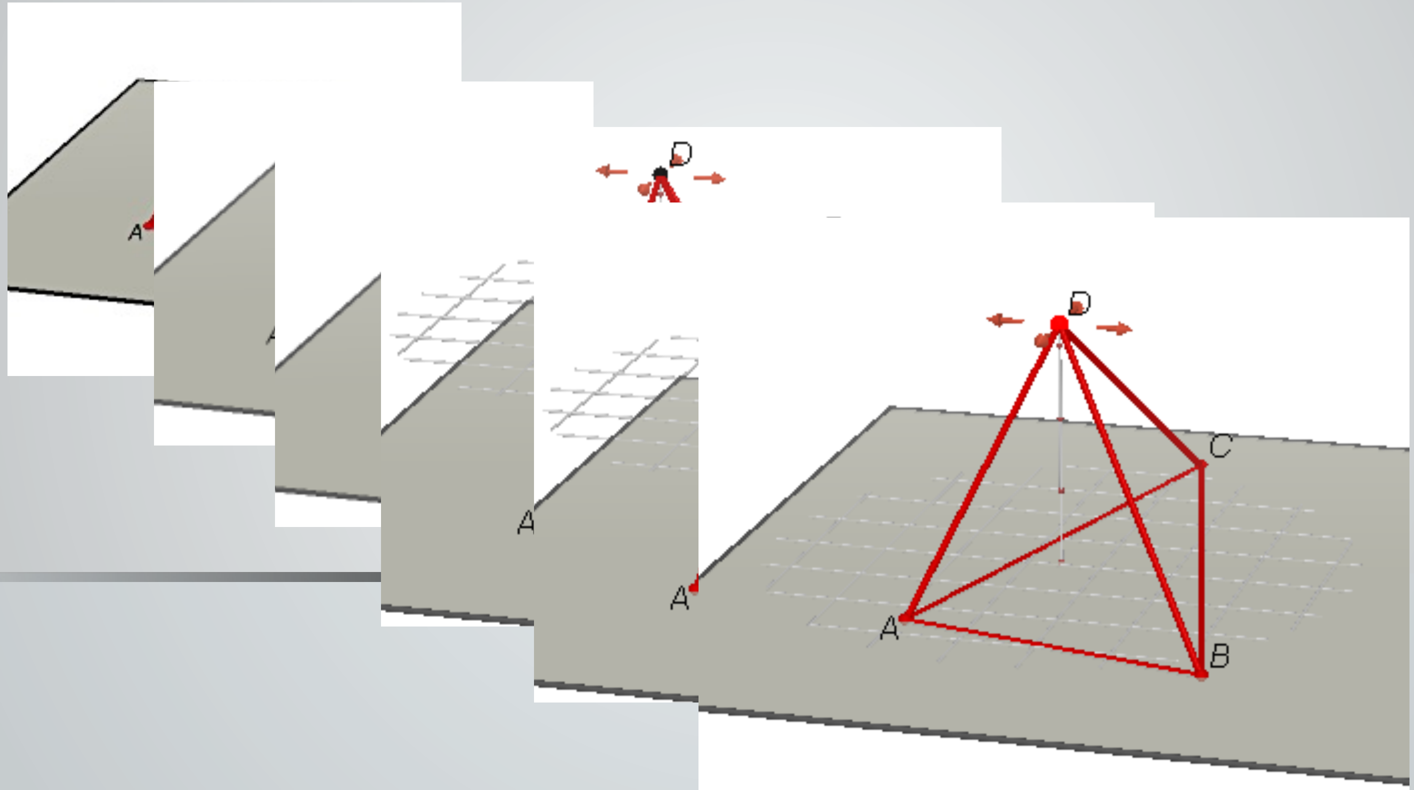

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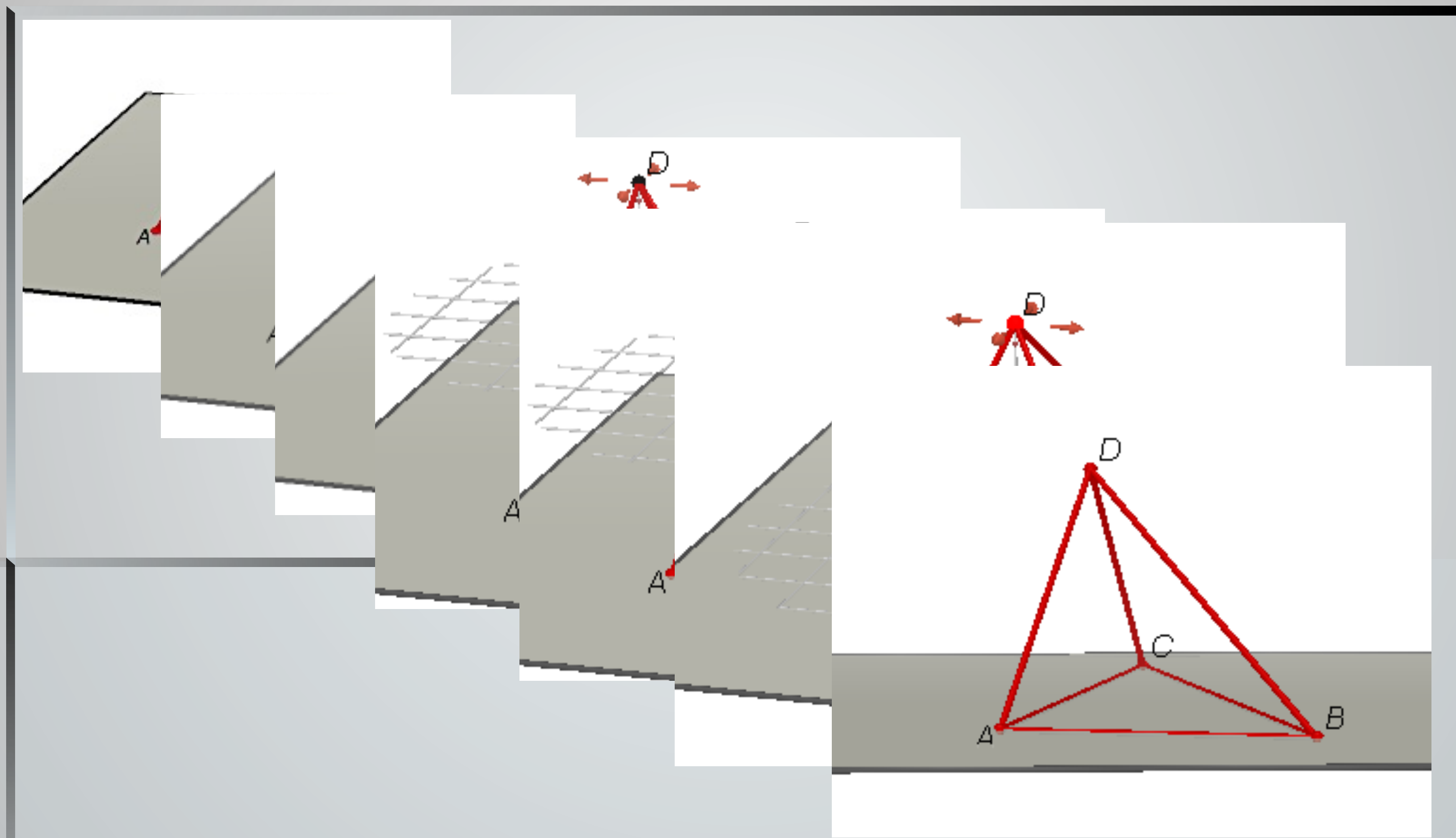


M & S play around with the redefinition of one vertex

M: It [the quadrilateral] becomes a tetrahedron. (…)

T: *<T talks about the bimedians>* Do they continue to meet?

M, S: Ya.

T: Where did they meet? *<T reads what M and S wrote before "the bimedians meet in a point H. H divides the bimedians into two equal parts">* Does this still happen?

S: Hmm, at sight it seems to do, yea *<M and S looks at the figure on the screen>*

Story 2

M & S play around with the redefinition of one vertex

M: It [the quadrilateral] <u>becomes a tetrahedron</u>. (…)

T: *dians>* Do they <u>continue</u> to

Seeing something *AS* something else

M,

T: Where did they meet? *<T reads what M and S*

What makes you so sure that mathematical logic corresponds to the way we think? Logic formalizes only very few of the processes by which we actually think. The time has come to enrich formal logic by adding to it some other fundamental notions. What is it that you see when you see? You see an object ***as*** *a key, a man in a car* ***as*** *a passenger, some sheets of paper* ***as*** *a book. It is the word 'as' that must be mathematically formalized…*

(Rota vs. Ulam, in Hofstadter, 2005)

M & S explore a quadrilateral and its medians with Cabri

S: It's <u>the same thing</u>.
M: Ya, it is.
S: It's <u>always</u> upside-down. <*S refers to the tetrahedron with vertices the centroids of the faces that they have constructed*>
M: "A" corresponds to "A'", and the others as well. [being A one vertex and A' the centroid of the opposite face] The <u>same properties</u> hold.

Story 3

M & S explore a quadrilateral and its medians with Cabri

S: It's <u>the same thing</u>.

M:

Seeing *AS* the same thing

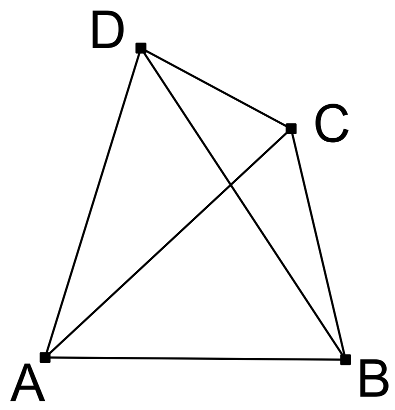
S: . *<S refers to the* *...s the centroids of the faces that they have constructed>*

M: "A" corresponds to "A'", and the others as well. [being A one vertex and A' the centroid of the opposite face] The <u>same properties</u> hold.

Story 3

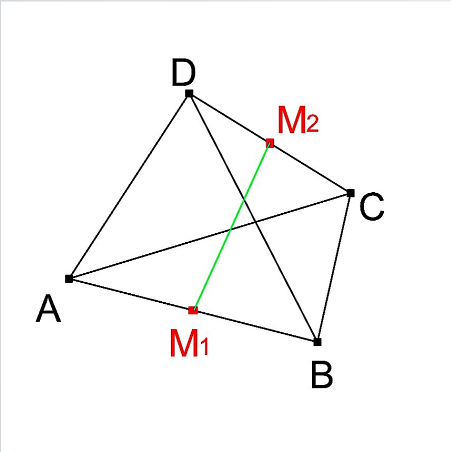
“The same thing”

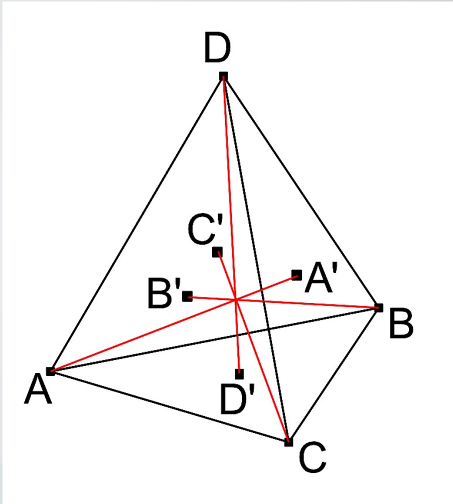
four *vertices*
six *edges*
four *faces*



four *vertices*
six *edges*
four *faces*

“The same properties”





The students were given one final Cabri 3D file containing two figures that seemed to be exactly the same

T: What are the figures? What do you see?

E: They're pyramids because we're using Cabri 3D.

C: No! The one on the left is a quadrilateral, the one on the right a pyramid.

T: How do you know?

C: I dragged the vertices, there are no projections on the left, yea on the right.

S: <*T asks "What about you?"*> The same, but we used the *Glassball*.

T: They might seem the same object, but they are not. We have seen that the same definitions and properties hold for both figures. So, does it actually matter what they are?

Ss: No!

Story 4

The students were given one final Cabri 3D file containing two figures that seemed to be exactly the same

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T: How do you know?

C: I dragged the vertices, there are no projections on the left, yea on the right.

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The students were given one final Cabri 3D file containing two figures that seemed to be exactly the same

T: What are the figures? What do you see?

E: They're pyramids because <u>we're using Cabri 3D</u>.

C: <u>No!</u> The one … the one on the right a p…

Dragging & Glassball

T: How do you …

C: I <u>dragged</u> the vertices, there are no <u>projections</u> on the left, yea on the right.

S: *<T asks "What about you?">* The same, but we used the *<u>Glassball</u>*.

T: They might seem the same object, but they are not. We have seen that the same definitions and properties hold for both figures. So, <u>does it actually matter what they are</u>?

Ss: No!

Story 4

Final remarks

- Seeing in 3D involved **visual challenges** that had mainly to do with conflicts between knowing and seeing and with the perception of the third dimension in flat diagrams

- The students needed to take on **multiple perspectives**, as if they were taking on various physical positions from which to see a figure, as bodily projecting themselves both beyond and around it

Final remarks

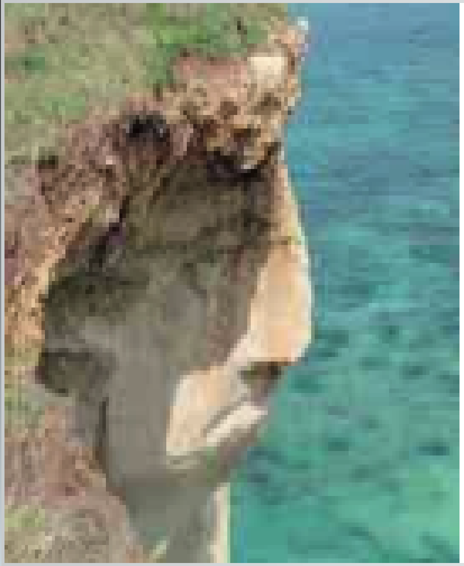
- The DGS fostered visual engagements of the students

- The students were engaged in dynamic visual experiences with the diagrams, effecting **new kinds of vision** that pushed them towards a search for similarities and differences, invariants and changes, between quadrilaterals and tetrahedra

Final remarks

- More confidence of the students in seeing in 3D
- Emergence of two different geometrical figures from a single diagram
  - discerning relevant elements
  - changing perspective
  - making visible features that were not present at a first glance

What do you see?



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