ZDM Mathematics Education
DOI 10.1007/s11858-009-0220-6

ORIGINAL ARTICLE

Concrete models and dynamic instruments as early technology tools in classrooms at the dawn of ICMI: from Felix Klein to present applications in mathematics classrooms in different parts of the world

Maria G. Bartolini Bussi · Daina Taimina · Masami Isoda

Accepted: 18 October 2009

Abstract Most national curricula for both primary and secondary grades encourage the active involvement of learners through the manipulation of materials (either concrete models or dynamic instruments). This trend is rooted in the emphasis given, at the dawn of ICMI, to what might be called an experimental approach: the links between mathematics, natural sciences and technology were in the foreground in the early documents of ICMI and also in the papers of its first president, Felix Klein. However, the presence of this perspective in teaching practice is uneven. In this paper, we shall reconstruct first an outline of what happened in three different parts of the world (Europe, USA and Japan) under the direct influence of Klein. Then, we shall report classroom activities realized in the same regions in three different research centres: the Laboratory of Mathematical Machines at the University of Modena and Reggio Emilia, Italy (http://www.mmlab.unimore.it); the pedagogical space of Kinematical Model for Design Digital Library at Cornell, USA (http://kmoddl.library.cornell.edu/); and the Centre for Research on International Cooperation in Educational Development at Tsukuba University, Japan (http://math-info.criced.tsukuba.ac.jp/). They have maintained the reference to concrete materials (either models or instruments), with original interpretations that take advantage of the different cultural conditions. Although in all cases the reference to history is deep and systematic, the synergy with mathematical modelling and with information and communication technologies has been exploited, not to substitute but to complement the advantages of the direct manipulations.

Keywords Models · Instruments · ICT · ICMI · Felix Klein

M. G. Bartolini Bussi (✉)
Dipartimento di Matematica, Università di Modena e Reggio Emilia, Via Campi 213/B, Modena, Italy
e-mail: bartolini@unimore.it;
mariagiuseppina.bartolini@unimore.it
URL: http://www.mmlab.unimore.it

D. Taimina
Department of Mathematics, Cornell University, Ithaca, NY 14853-4201, USA
e-mail: daina.taimina@cornell.edu; dtaimina@lanet.lv
URL: http://www.math.cornell.edu/~dtaimina/

M. Isoda
Graduate School of Human Comprehensive Science, University of Tsukuba, 305-8572 Tsukuba, Japan
e-mail: msisoda@human.tsukuba.ac.jp;
msisoda@muc.biglobe.ne.jp; isoda@criced.tsukuba.ac.jp
URL: http://math-info.criced.tsukuba.ac.jp/

1 Introduction

The relationships in teaching between mathematics and the concrete materials of the real world have a long history that does not suit the space of a single paper. We shall limit ourselves to some classical quotations (Castelnuovo, 2008). Jan Comenius (1592–1670) defended in Didactica Magna (1657) the relevance of the manipulation of concrete things in every individual experience of knowledge construction:

> "Everything must be presented to the senses as much as possible; to wit, the visible to the eye, the audible to the ear, odors to the sense of smell, the tastable to the taste, and the touchable to the sense of touch; and, whenever something can be grasped by more than one sense at one time, let it be presented to them at one time. One may, however, if the things themselves

Published online: 07 November 2009

> cannot be presented, use representations of them, such as models and pictures. […] It is a mistake to let rules in an abstract form go before, and afterwards explain them in examples. For the light must go before him for whom it is intended to shine. […] Whatever is to be done, must be learned by doing it. Mechanics do not detain their apprentices for a long time with meditations: they put them to work at once, that they may learn to forge by forging, to carve by carving, to paint by painting, etc. So the pupils should also learn at school to write by writing, to speak by speaking, to count by counting, etc. Then the schools are workshops filled with the sound of work" (Comenius, 1657).

These general methodological rules were, at the time of Comenius, largely ahead of the teaching practice, as (at least in Europe) there were no public educational institutions for everybody. One century later in France, another voice was raised, with explicit reference to the teaching of geometry linked with 'reality' (whatever that means). Alexis Clairaut (1713–1765) discussed the issue as follows:

> "Some authors put after each basic statement (of geometry) its practical use: yet in this way they establish the usefulness of geometry, without making geometry learning easier. Because if any statement is before its use, the mind can come at sensible ideas only after having struggled with abstract ideas" (Clairaut, 1741).

He then suggested his method to evade the above drawbacks:

"I have planned to find all that could have given rise to geometry; and I have managed to explain its principles as naturally as possible, like early inventors" (Clairaut, 1741). Hence, he claimed that in geometry teaching it is necessary to start from measuring land (the Greek etymology of "geometry": measurement of earth or land).

These educational principles were available, although neither widely shared nor widely applied, when, for the first time in Europe, in the first article of the France National Constitution (1791), the right to a system of public instruction was stated: "Public instruction for all citizens, free of charge in those branches of education which are indispensable to all men."

In the first half of the nineteenth century, a German educator, Friedrich Froebel (1782–1852), launched a practice of active methods for children. He designed open-ended instructional materials called the gifts, with complementary occupations (Le Blanc, n.d.). These were for use both in kindergarten and in school, and gave children hands-on involvement in practical learning experiences through play. Foundational to the development of the gifts was the recognition of the value of playing with blocks. Through proper use of the gifts, the child progresses from the material to the abstract: from the volumetric lessons offered by blocks, through the two-dimensional planar ones elucidated by play with parquetry tiles (flat, geometrically patterned wooden shapes), to deductions of a linear nature drawn from stick laying and to the use of the point in pin-prick drawings. Points, in turn, describe a line, and the child completes the logic by returning from 2D to the 3D realm of volume through joining small malleable peas with toothpicks and onto solid three-dimensional work in clay.

Although Froebel's work was mainly oriented to young children, his approach was representative of an atmosphere shared with contemporaneous mathematicians in Central Europe. For instance, Gaspard Monge (1746–1818), a mathematician deeply involved in French revolution, supported the creation of beautiful models of surfaces.

> "Monge is known as the father of differential geometry and his efforts in the early 1800s to classify surfaces by the motions of lines, along with his descriptive geometry for representing three-dimensional surfaces in two-dimensions, led naturally to the construction of elaborate models made of tightly stretched strings. One of his students, Theodore Olivier (1793–1853), built some of the most beautiful concrete models of mathematical concepts ever made. He also made some money in the process: the models were expensive. Olivier sold them to the emerging technical schools in the United States, which were attempting to emulate the example of Monge and the Ecole Polytechnic." (Mueller, 2001).

It is in this atmosphere that one must situate the work of Felix Klein (1849–1925) and the other European founders of ICMI.

This paper is divided into two parts organized around two main issues:

- First part: the influence of Felix Klein at the dawn of ICMI in different parts of the world (Europe, USA, Japan).
- Second part: present applications in today's classrooms of the above seminal ideas in the same parts of the world.

The former concerns the historical roots; the latter concerns the present utilization of concrete models and dynamic instruments, which are complementing yet neither contrasting nor excluding the recourse to information and communication technologies.

2 First part: concrete models and dynamic instruments at the dawn of ICMI

2.1 The European roots and the contribution of Felix Klein

The use of concrete models and dynamic instruments was common in Europe in the seventeenth and eighteenth centuries (see for instance, Maclaurin 1720), but it had a new impulse in the nineteenth century. In particular, the second half of the nineteenth century was a time when many new mathematical ideas were born, combining together previously separated parts of mathematics. Mathematicians began to build intricate models out of wood, string and plaster. In Germany, the main advocate for the use of concrete models and dynamic instruments was Klein. In many ways, Monge in France and Klein in Germany set the standards for how mathematics was taught in Europe, Northern America and the Far East in the nineteenth and early twentieth centuries (Klein & Riecke, 1904; see also Schubring, 1989). One of the most thorough approaches in mathematics, to use concrete models and dynamic instruments in education and research, is the famous collection of models in Göttingen. This model collection already had a long history when Felix Klein and Hermann Amandus Schwarz (1843–1921) took over the direction of the collection. Especially under the direction of Klein, the collection was systematically modernized and organized for the education of students in geometry and geodesy. The first clear indication of an interest in model building had appeared in the "Monthly reports of the Royal Prussian Academy of Science in Berlin" in 1873, where it was described that Ernst Eduard Kummer (1810–1893) presented a plaster model of the Steiner surface, which he had constructed himself. A wide production of models began in the 1870s, when Felix Klein and Alexander Brill (1842–1935) founded a laboratory for the construction of models and instruments at the Munich Technische Hochschule. The production and detailed study of models was one of the purposes of the problem sessions directed by Klein and Brill at the Royal Technical University in Munich. Some of the models, which were really constructed only as exercises or examples, showed that these types of visual aids were not at all superfluous; on the contrary, they were of great value (Schilling, 1911).

Klein required his students to make models in connection with their dissertations on algebraic surfaces. Klein, Brill and their students built a number of the models that ultimately became a part of the collection marketed by Brill's brother Ludwig, the owner of a publishing firm in Darmstadt. When Ludwig Brill took over the sales of the models, they were put together in series, each being accompanied by a mathematical explanation. Other producers also offered models, devices, and instruments for mathematics, physics and mechanics. In 1892, under contract of the newly established German Mathematical Society, Walter von Dyck (1856–1934) assembled a complete catalogue of such products (Fischer, 1986). He was one of the creators of the Deutsches Museum of Natural Science and Technology in Munich, and he was also appointed as the second director of the museum in 1906. The Deutsches Museum was the first of its kind and its ideas were soon copied by other science museums around the world.

The Munich collection was considered so important that later Klein exhibited the models on the occasion of the World's Columbian Exposition 1893 in Chicago. Models from the collection of L. Brill were later purchased by many mathematics departments throughout Europe and the USA (Parshall & Rowe, 1991).

Later, Klein, as the first president of the International Commission on Mathematical Instruction, also supported in secondary schools in a very strong way the recourse to models, instruments and practical work. He reaffirmed the importance of work with models and instruments also in his books: *Elementary mathematics from an advanced standpoint* (Klein, 1924, 1925), the famous series for secondary mathematics teachers, to "put before the teacher, as well as the maturing student, from the view-point of modern science, but in a manner as simple, stimulating and convincing as possible, both the content and the foundations of the topics of instruction, with due regard for the current methods of teaching (Klein, 1908, preface to the first edition, published in Klein, 1924, p. III)."

The influence of the title of this series was large: in Germany, courses were opened for prospective teachers with this exact name; the same name is still used in Italian universities for the epistemological courses in the education of prospective mathematics teacher.

Just to give an example of Klein's approach, we may quote what he wrote about the practice in calculating with integers (Klein 1924), after having described in detail the famous mechanical calculator Brunsviga (see Fig. 1).

> "Let us consider for a moment the general significance of the fact that there really are such calculating machines, […]. In the existence of such a machine we see an outright confirmation that the rules of operation alone, and not the meaning of the numbers themselves, are of importance in calculating, for it is only these that the machine can follow; it is constructed to do just that, it could not possibly have an intuitive appreciation of the meaning of the numbers. […] Although it is not historically authenticated, still I like to assume that when Leibniz invented the calculating machine, he not only followed a useful

Fig. 1 Brunsviga calculating machine (courtesy of New Beginning Antique at http://www.newbegin.com)

purpose, but that he also wished to exhibit, clearly, the purely formal character of mathematical calculation."

For this very reason, Klein wished that every teacher of mathematics should become familiar with calculating machines, and that it ought to be possible to have it demonstrated in secondary instruction. In this case, Klein defended the recourse to mechanical instruments as a means to become familiar with the very abstract distinction between the syntactic and the semantic aspects in mathematics.

Several dynamic instruments were discussed also in the second volume on Geometry (Klein, 1925): for instance, a polar planimeter (p. 14), a mechanism to perform affine transformations (p. 75), a linkage to realize a circular inversion and, hence, to guide a point so that it will describe a straight line (p. 100). This last instrument, called, after Charles-Nicolas Peaucellier (1832–1913), the Peaucellier straight line mechanism, is represented in Fig. 2.

In the same years, Franz Reuleaux (1829–1905), a professor of mechanical engineering in Berlin, built a large collection of 800 mechanical models in Berlin and marketed 350 of them to universities around the world. Unfortunately, much of these collections were destroyed during World War II, but some originals and reproductions of these models can be found in the Deutsche Museum in Munich, the University of Hannover, Kyoto University, Moscow's Bauman Technical School, Karlov University in Prague and possibly in some other places we do not know yet. The largest collection of these models is in Cornell University where there are 220 (from the originally acquired 266) Reuleaux models (see Fig. 3). We shall illustrate this point later.

Reuleaux believed that there were scientific principles behind the invention and the creation of new machines,

Fig. 2 Peaucellier straight line mechanism (http://KMODDL.library.cornell.edu)

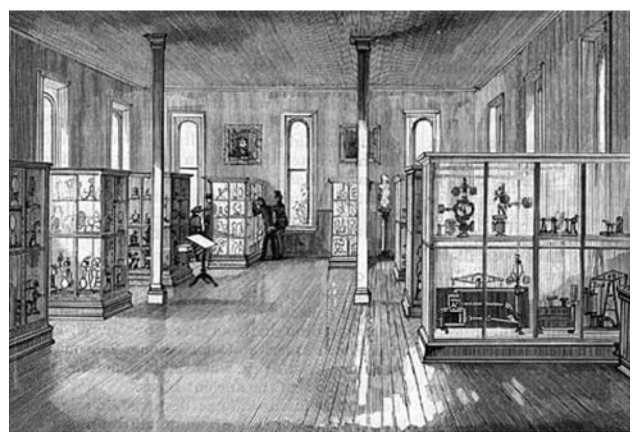

Fig. 3 Reauleaux kinematic model display in Sibley Hall, Cornell University in 1887. *Scientific American* cover

what we call "synthesis" today. This belief in the primacy of scientific principles in the theory and design of machines became the hallmark of his worldwide reputation, especially in the subject of machine kinematics (Moon, 2002). Hence, Reuleaux was a champion of the application of mathematics. Reuleaux also devoted serious attention to education and the role of mathematics:

"The forces of nature which advance taught us to look to for service are mechanical, physical and chemical; but the prerequisite to their utilization was a full equipment of mathematics and natural sciences. This entire apparatus we now apply, so to say, as a

> privilege. […] The instruction in the polytechnic school has of necessity to adopt as fundamental principles the three natural sciences—mechanics, physics, and chemistry, and the all-measuring master art of mathematics" (Reuleaux, 1876).

Franz Reuleaux incorporated mathematics into design and invention of machines in his work *Kinematics of machinery*. For mathematicians, he is best known for the Reuleaux triangle, which is one of the curves of constant width (see Fig. 4.) This curved triangle can be seen in some gothic windows; it also appears in some drawings of Leonardo da Vinci (1452–1519) and Leonhard Euler (1707–1783), but Reuleaux in his *Kinematics* gave the first applications and complete analysis of such triangles, and he also noticed that similar constant-width curves could be generated from any regular polygon with an odd number of sides. (Taimina & Henderson, 2005).

Reuleaux classified his mechanisms using an alphabet, that is, assigning letters to different groups of his mechanisms. In that way, he stressed that each individual mechanism was like a letter in an alphabet and, on combining them together, we would get words and sentences, which denote machines. His style of classification resembles later classification ideas used in topology and theoretical computer science. The largest number (39) of Reuleaux mechanisms is in the so-called S-series: straight line mechanisms (see Fig. 3) (http://kmoddl.library.cornell.edu/model.php?cat=S). Changing circular motion into straight line motion had been a challenge to technology since ancient times (Kempe, 1877). This problem was crucial to James Watt (1736–1819) when he was working on improving the steam engine (Taimina, 2005b; Henderson & Taimina, 2005b).

Fig. 4 Reuleaux triangle rotating in the square (photo Prof. F. Moon)

Klein defended the importance of models and instruments to illustrate a theory, against the disposition of pure mathematicians. In the same years, these ideas were shared by the officers of ICMI. At the fifth International Congress of Mathematicians in 1912, the section on didactics received ICMI reports examining the state and trends of mathematics teaching. In particular, the report of the Sub-Commission A (mathematics in secondary education) focused on "Intuition and experiment in mathematical teaching in secondary schools" (Smith, 1913). Eugene David Smith (1860–1944) discussed "contemporary developments aimed at providing an 'intuitive', 'perceptual', 'experiential' and 'experimental' base for the subject (p. 611), through 'applying mathematics seriously to the problems of life, and… visualizing the work' (p. 615). This, then, represented a foundational theme for the ICMI. […]. The main areas which the Sub-Commission singled out […] were geometrical drawing, graphical methods, practical measuring and numerical computation" (Ruthven, 2008). Further details may be found in Giacardi (2008).

2.2 Klein's influence on mathematics education in the USA: the Cornell collection

In 1857, the US Military Academy ordered 26 Olivier models, of which the Department of Mathematical Sciences still has 24 (Shell-Gellasch & Acheson, 2007). In this way, the first common schools attempted to emulate the "objective" practices of their German counterparts. Sets of geometric solids were sold to the new schools with claims that they would help teachers present a common curriculum. School boards and government commissions formalized the arrangement, making models a required component of mathematics instruction in many states. Business was so good during the nineteenth century that model makers were able to diversify into much more lucrative catalogues of Mathematical Apparatus, which included everything from the latest in "noiseless" drawing slates and elegant "pointing rods" to hand-crafted "numerical frames," made of the finest woods. There was eventually a backlash, however, as teachers began to complain that the expensive, and increasingly complicated, apparatus was driving the curriculum. It is also of interest to look at the response to the educational "technology" of the time, i.e. concrete models in American colleges. The use of concrete models in the classroom caused the same kind of divisive debates that surround modern visualization technology. Then, as in the present, usage varied from place to place and from instructor to instructor. The most enthusiastic endorsements in the report come from teachers who used the models to help students visualize problems in three-dimensional calculus and the "higher surfaces" of descriptive geometry (Mueller, 2001).

Although Klein declined several opportunities to teach in the USA, he nevertheless had a long-lasting influence on American mathematics. The terrain was fertile, as Cornell University in 1882 had acquired a collection of mechanical models designed by Reuleux (see Fig. 2). The first Chair of the Cornell Mathematics Department, James Edward Oliver (1829–1895), went to study mathematical physics in Cambridge; but, after hearing enthusiastic accounts of Klein's lectures, Oliver wrote to Klein from Cambridge to ask whether it would be possible for him to spend some time in Göttingen. One of the things in which he was particularly interested was "to get pretty fully Klein's ideas as to methods of teaching, topics and courses of study, and promising directions for original research by my young men". Hence Cornell emerged as a prime sphere of Klein's influence in the USA (Parshall & Rowe 1991). Oliver sent his student, Virgil Snyder (1869–1950), to study with Klein in Göttingen. After the World's Fair in Chicago (1893) and the famous Evanston lectures, Klein travelled to the USA, and one of the points of his interest was Cornell University, where he was hosted by Oliver (Parshall & Rowe 1991).

As mentioned earlier, Cornell hosted the famous Reauleaux collection, and hence the attitude towards experimental approaches was very well established. We consider later the pedagogical revival of this collection in the twenty first century.

2.3 Klein's influence on mathematics education in Japan: mathematization in curricula

Japan was under the influence of Chinese mathematics until the sixteenth century; the influence of European mathematics started in the middle of the nineteenth century from the UK, France, Germany and USA (Isoda, 2004). To develop new academy and school mathematics, some mathematicians and mathematics educators studied mathematics in Germany at the end of the nineteenth century and at the beginning of the twentieth century. When they came back, they became engaged in developing an integrated curriculum as part of the Klein movement using the ideas such as in Perry (1913) and Sanden (1914). This movement resulted in the establishment in 1919 of the Japan Society of Mathematical Education for the improvement of secondary school mathematics. Several textbooks based on this movement had been published, such as those by Kuroda (1920), and several contents had been experimentally taught at the secondary school attached to the Tokyo higher normal school (origin of the University of Tsukuba) with respect to calculus (see Sanden 1914; Kuroda 1927) and projective geometry. Yet, the integration into curricula was postponed because of the earthquake, which had burned the capital Tokyo in 1923. After the earthquake, the integration of the mathematics curriculum was completed during WWII. It was done in the curriculum reform of 1942 and published as textbooks in 1943 with the key word of mathematization (*sugakuka* in Japanese). In the textbooks, a number of mechanical instruments were treated as the subject matter for mathematization. The textbooks' style had a workbook format with open-ended problems enabling students to learn by themselves with the support of their teachers during the air alerts of the war. At first, the textbooks focused on the construction of mental objects, which should be mathematized; later, it became more sophisticated mathematically, repeatedly on a spiral sequence: similar mathematical situations were explored again and again for developing mental object in mathematics.

Figure 5 represents an example of mathematization related to the design of the cap of an electric lamp. In grade 6 (11-year-old students), students explore how to draw the development of the section of cylinder experimentally by using a set of different viewpoints such as front, side and top views and by rotating the viewpoints on the radius of the bottom part of the circle for their drawing. Here, students study the methods for analysing solids by projection onto planes. Later, the same situation is re-explored for studying conic sections in grades 9 and 10. The same mental object, which the situation explored, will be re-explored in grade 11, with the recourse also to trigonometric functions.

Another example is taken from the 1943 textbooks (approved by Monbusyo, 1943) for 14-year olds with the aim of realizing mathematization with open-ended approach from the situation to elementary geometry and from elementary geometry to analytic geometry. There

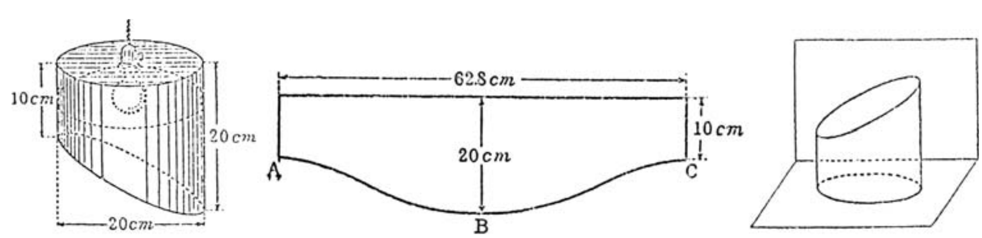

Fig. 5 The first step of designing the cap of an electric lamp (Monbusyo, 1943)

were designed shifts from everyday mechanisms (Fig. 6) to the geometrically simple mathematical instruments (Fig. 7), then to geometrically deduced mathematical instruments (Fig. 8) and finally to algebraic representations (all the figures are taken from Monbusyo, 1943). This sequence beginning from Fig. 7 is the same as in the textbook by Franz van Schooten (1615–1660), *De organica Conicarum Sectionum Constructione* (1646), which first treated conics plane geometrically and algebraically, instead of as the section of a cone.

Mechanical instruments were well integrated in the textbooks around World War II, but were gradually lost after the war, and only the pantograph has remained after the modernization of the 1960s–1970s curriculum because of the introduction of more algebraic approaches such as linear algebra.

2.4 Klein's influence on mathematics education in Italy: the collections in mathematical institutes

Italian mathematicians shared Klein's interest in activities with concrete models and dynamic instruments. Between the nineteenth and twentieth centuries, many models and instruments were available in nearly all the universities, at the mathematical institutes, where the education of both professional mathematicians and secondary mathematics teachers took place an example is shown in the Fig. 9. Still, in the early decades of the twentieth century, the importance of the reference to intuition (developed from the consideration of models) was a feature of the so-called Italian school of algebraic geometry. Guido Castelnuovo (1865–1952) presented their work in this way:

> "We had created (in an abstract sense, of course) a large number of models of surfaces in our space or in higher spaces; and we had split these models, so to speak, between two display windows. One contained regular surfaces for which everything proceeded as it would in the best of all possible worlds; analogy allowed the most salient properties of plane curves to

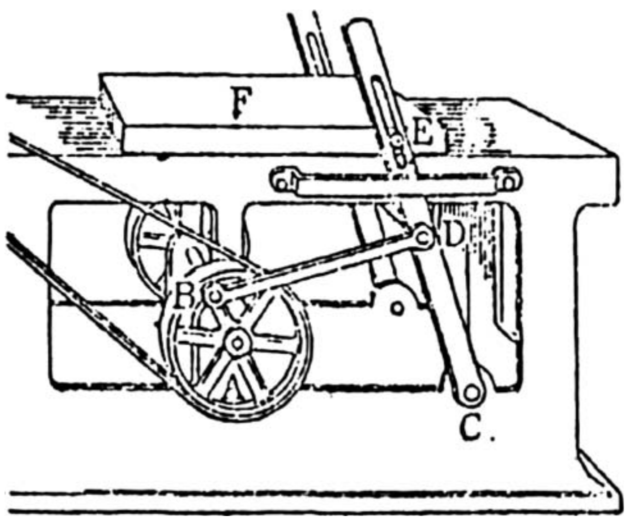

Fig. 6 How does the point F move? (Monbusyo, 1943, vol. 3, p. 2)

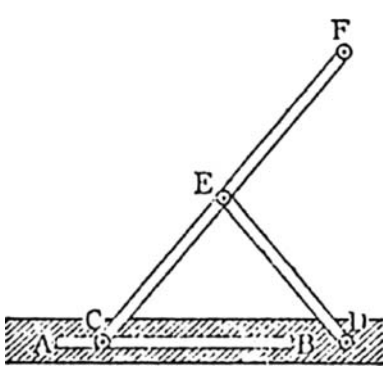

Fig. 7 The locus of the point F (Monbusyo, 1943, vol. 3, p. 2)

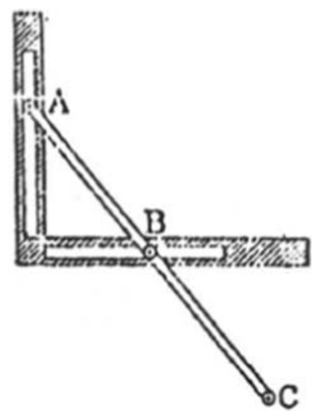

Fig. 8 How does the point C move? (Monbusyo, 1943, vol. 3, p. 2)

Fig. 9 A plaster model of Steiner surface (courtesy of G. Ferrarese at the Dipartimento di Matematica, Università di Torino)

> be transferred to these. When, however, we tried to check these properties on the surfaces in the other display, that is on the irregular ones, our troubles began, and exceptions of all kinds would crop up… With the aforementioned procedure, which can be likened to the type used in experimental sciences, we managed to establish some distinctive characters between the two surface families" (Castelnuovo, 1928) (http://www.icmihistory.unito.it/portrait/castelnuovo.php).

Some decades later, Luigi Campedelli (1903–1978) sharply commented that a reader could have read the above vivid description of concrete models, ignoring the parenthesis "in an abstract sense, of course" (Campedelli, 1958).

In the following decades, under the influence of the Bourbaki's trend, the collections of models fell into neglect, were warehoused and even destroyed. Recently, the remains of some of the collections have been restored and arranged in display cases as relics of the past. A complete catalogue of the still existing collections of these kinds of models and instruments has been prepared by Palladino (n.d.), who visited all the departments of mathematics in Italy to check for their existence and present condition. We shall consider later the use of these kinds of instruments and models in today's classrooms.

2.5 Different approaches

These three examples show different approaches to the complex relationships between mathematics as a mental activity and the exploration of the concrete materials of the real world. To sum up, as a first approximation, one may compare the real world on the one hand and the mathematical world on the other hand. Klein, as evidenced by the above quotation of Leibniz' intentions, tends to consider concrete models and dynamic instruments as representations of mathematical concepts and processes. They can also be used to solve problems in the real world, but this does not seem to be the main issue. Rather concrete and dynamic exploration of models and instruments (realized in the real world) may be useful for teachers, at all school levels, to foster both mathematical understanding and the production of conjectures. This is evident also in the above quotation from Castelnuovo (1928) regarding collections in the mathematical institutes, the purpose of which is to train both professional mathematicians and mathematics teachers.

On the contrary, Reuleaux (1876) championed the importance of applied mathematics, with emphasis on the interaction between mathematics, mechanics, physics and chemistry.

Japanese mathematicians and mathematics educators in the early decades of the twentieth century have chosen an integration between the two approaches, which are both represented: the cultural approach, championed by Klein, with reference to the history of mathematical ideas, and the application approach (mathematization) with reference to the mathematical modelling of concrete instruments. In principle, they are not incompatible with each other; rather they show the complexity of relationships between mathematics and the world of concrete experience. Gabriel Koenigs (1858–1931) expressed well the complex links between pure and applied mathematics, referring to a pantograph for homotheties.

> "The theory of linkages is supposed to start in 1864. Surely linkages were used also earlier: a dedicated and precise scholar might track down them in the most ancient times. One might discover in this way that each age has in hand, so to say, yet without awareness, the discoveries of future ages: the history of things often anticipates the history of ideas. When in 1631 Scheiner published for the first time the description of his pantograph, he certainly did not know the general concepts contained as germs in his small instrument; we claim that he could not know them, as they are linked to the theory of geometric transformations, that is a theory typical of our century and gives a unitary stamp to all the made advances" (Koenigs, 1897).

3 Second part

3.1 Use today in mathematics classrooms

In the first part of this study, we have reported some historical information about one of the richest periods in the recent history of mathematics education, as far as the relationships between pure and applied mathematics are concerned, in three different continents (Europe, America and Asia). This reconstruction meets the needs of a historical issue about resources and technologies in the International Commission on Mathematical Instruction. Yet, it would be misleading to avoid reporting about the present use of concrete models and dynamic instruments described above in the same countries, where specific research centres have been continued or created anew. Actually, the historical traditions dating back to Klein and other scholars have been resumed in three different places, where the three authors of this paper developed important programs in mathematics education involving teachers and classrooms.

3.2 Use today in mathematics classrooms: KMODDL pedagogical space at Cornell University (USA)

Taking advantage of the Reuleaux collection, in 2002, the Cornell University started to develop KMODLL (kinematic models for design digital library). Now these models can be explored on the Web site http://kmoddl.library.cornell.edu/ or http://kmoddl.org. On the Web sites, besides having still images of models, there are historical information and interactive movies that allow a viewer to explore how these models work. The Web site also has scanned rare books that are important in the history of technology. A significant part of this project works in connecting concrete models and dynamic instruments, on the one hand, and mathematical ideas behind these, on the other hand, for the purpose of using them in the classroom. Teaching materials have been developed and can be found in the section of the Web site called Tutorials. There are also stereo lithographic files that allow 3D printing of some of the models. This material is available for teachers who wish to introduce these activities in their classroom.

Cornell Faculty in Mechanical Engineering, Mathematics and Architecture are using the KMODDL Web site in the classroom to teach mathematical principles of mechanisms as well as machine design and drawing. Mathematical ideas from this collection have found their place in the geometry textbook (Henderson & Taimina, 2005a).

The initial evaluation of the KMODDL in an undergraduate mathematics class has confirmed the usefulness of various physical and digital models in facilitating learning, and revealed interesting relationships among usability, learning and subjective experiences of the students (Pan et al. 2004).

Reuleaux models have been also the object of a 9-week interdisciplinary project 'Exploring Machine Motion Design' in area schools for grades 7–9, carried out several times in the last few years. The choice of exploring kinematic models was determined by the possibility to take actual Reuleaux models into the schools. During this project, students learned about the history of engineering and the role of mathematicians in it. One of human's oldest mechanical devices is gears. The earliest written records on gearing are dated from about 330 BC in the writings of Aristotle. He explained gear wheel drives in windlasses, pointing out that the direction of rotation is reversed when one gear wheel drives another. The most probable uses were in clocks, temple devices and water lifting equipment. The Romans and Greeks made wide use of gearing in clocks and astronomical devices. Gears were also used to measure distance or speed. One of the most interesting relics of antiquity is the Antikythera machine, which is an astronomical computer. Mathematical studies of gears were begun by Nicholas of Cusa (1401–1464) who, around 1450, studied cycloids. The famous painter Albrecht Dürer (1471–1528) also was interested in cycloids and he discovered epicycloids. The students in the project recalled that some of them played in their childhood with a toy called 'Spirograph'; such toys are still available in some art museum stores. Students were surprised to learn that one of the most important problems in the development of early technology was seemingly based on a simple question: how to draw a straight line (see Taimina, 2005b). This led the class discussion to some geometry of inversions and construction of linkages. Discussion on linkages was continued during a field trip to the Cornell Robotics Laboratory, where a group of researchers were working on designing evolutionary robots and testing these devices by asking the robots to re-create linkages in the kinematic model collection (see Taimina, 2005a). Exploring the history and mathematics of the universal joint helped students to see connections with spherical geometry that is a neglected topic in school curricula. At the end of this project, students were asked to create their own machine motion design, using as parts of their design models in the digital kinematic mechanism collection.

The 9-week interdisciplinary project could not cover all the riches of the kinematic model collection, so students submitted questions they had on the history of mechanisms and particularly on its connections with geometry. The goal of the project was to give students an opportunity to learn about basic elements of mechanisms, so that they could apply their knowledge to design their own machines. When this project was taught for the second time, students were able to use interactive computer design that involved elements from the historic kinematic model collection as building blocks for their own geometrical motion design.

3.3 Use today in mathematics classroom: CRICED at Tsukuba University (Japan)

As stated above, in Japan, the capacity of using mechanical instruments in the standard classroom has strongly diminished in the last decades. The use of mechanical instruments was rediscovered in 1990s to implement explorations within dynamic geometry software (DGS) inspired by the history of mathematics (for an example, see the following section). A project on mechanical materials such as LEGO (Isoda et al. 2001; Isoda & Matsuzaki, 2003) was established in 1996. Within this project, online teaching materials were developed (Isoda, 2008) in connection with Bartolini Bussi (1998), drawing on a number of historical textbooks. This project has influenced the revision of Japanese textbooks.

Isoda et al. (2006) developed the free software 'dbook' in xml format to produce e-textbooks including mathematical tools in classroom used together with an interactive

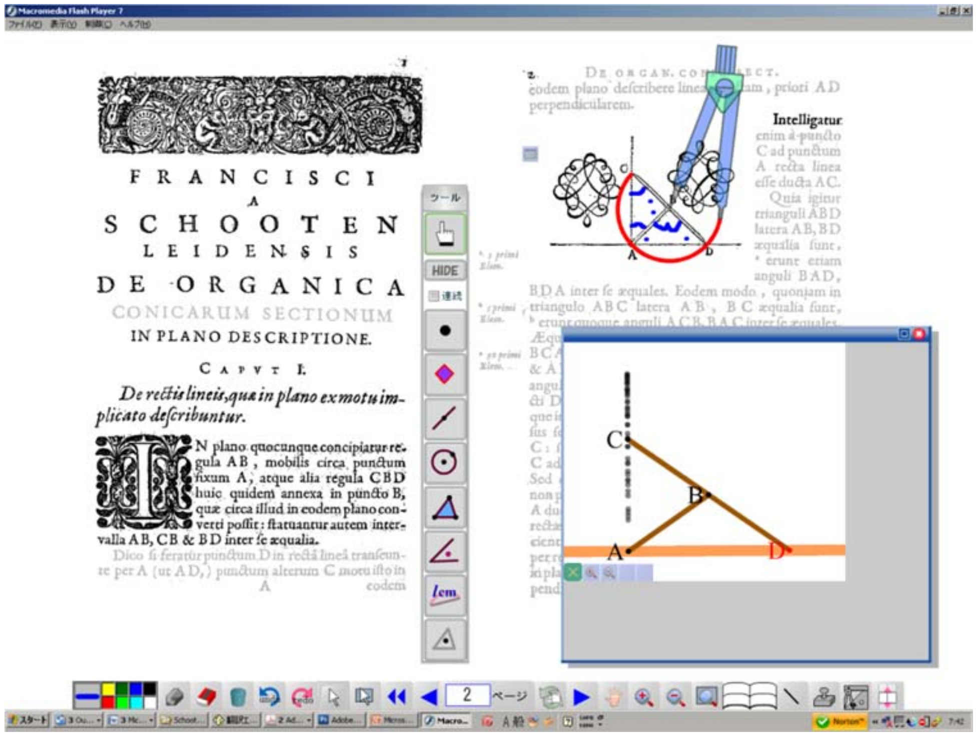

Fig. 10 The e-textbook developed from van Schooten (1646) by using dbook

board (Isoda, 2008). It enables us to use DGS and digital traditional instruments, such as a compass, on an e-textbook through Web sites while still keeping the traditional chalkboard classroom teaching approach (Fig. 10).

Figure 10 shows an e-textbook, which was developed from the textbook by van Schooten (1646). As shown by Isoda (2008), dbook was used for graduate students of the Federal University of Rio de Janeiro to find the intuition emerging in the activity with the textbooks. For example, the instrument in Fig. 10 (Fig. 7) is mechanically and mathematically the same as in Fig. 8, if we have geometrical intuitions. The participants understood well the existence of this intuition, which van Schooten had. But, without specific activity, students lost it when algebra got the upper hand over geometric intuition.

3.4 Use today in mathematics classroom: the Laboratory of Mathematical Machines in Modena (Italy)

In Italy, the importance of teaching the use of concrete models and dynamic instruments was defended by charismatic teachers such as Emma Castelnuovo (2008). Within this tradition, in the University of Modena and Reggio Emilia, a rich collection of more than 200 concrete models and dynamic instruments was constructed in a carpentry workshop, taking a leaf out of the phenomenology of geometry from the classical age to the nineteenth century. The collection is now stored in the Laboratory of Mathematical Machines (http://www.mmlab.unimore.it). It is always increasing, as new models and instruments are built every year. Briefly, a mathematical machine (concerning geometry) is a tool that forces a point to follow a trajectory or to be transformed according to a given law. Familiar examples of mathematical machines are the standard compass (that forces a point to go on a circular trajectory), the instruments described by Klein (1924, 1925) and reported above, the dynamic instruments of the Cornell collection and van Schooten's drawing devices used in the CRICED project.

The implementation of hands-on activities on mathematical machines, aiming at producing conjectures and constructing proofs about their functioning and mathematical properties, has been reported elsewhere (e.g. Bartolini Bussi & Pergola, 1996; Bartolini Bussi, 1998; Maschietto, 2005). There is also space for modelling activity, when dynamic geometry software is introduced (Bartolini Bussi, 2001).

Consider, for instance, the device for drawing a parabola by Bonaventura Cavalieri (1598–1647). In Chapter XLVI of *Lo specchio ustorio* (the burning mirror, Cavalieri, 1632), Cavalieri described a method for tracing a parabola by means of hard instruments, made up of rulers and set squares. In the following figures, the original drawing by Cavalieri (Fig. 11) is paired with a wooden copy of it (Fig. 12), shown in the same orientation to foster the identification of the same components. The tracing point is I (where a pencil lead R is put). The point A is fixed. The

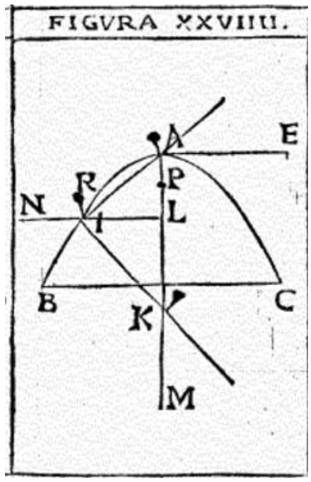

Fig. 11 The original drawing

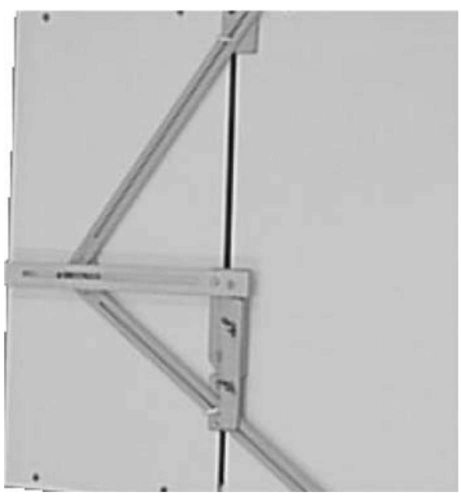

Fig. 12 A wooden copy

bar KL has a fixed length (evident in the wooden copy) and slides in the rail, dragging the set squares KLN and KIA, aAnd the ruler AI slides on the point A. During the motion, R (in I) traces an arch of parabola. In Cavalieri's drawing also the symmetric arch is drawn, in spite of the practical impossibility of drawing it, without modifying the set squares.

The wooden copy has been built in the Laboratory of Mathematical Machine drawing on Cavalieri's design. This copy is usually given to students (undergraduate prospective teachers), asking them to produce a digital model by means of Cabri software. Actually, by means of dynamic geometry software such as Cabri, it is possible to produce a construction that may be moved like the original to produce a parabolic arch. This is a true modelling task, where a concrete object is analysed to build its mathematical model, not described by equations but by analogical reproduction.

The process may be reported as follows:

First part: in the world of the dynamic instrument

(a) to look at the instrument; to catch the possibility of motion;
(b) to identify the fixed elements (the rail; the point A; the right angles KIA and KLN; the length KL), using, if necessary, measuring instruments;
(c) to understand that some elements are not essential in this kind of modelling (e.g. the rulers' thickness; the cracks in the bars; the screws).

Second part: in the geometrical world (*Cabri Geometry*)

(a) to draw (in the Cabri screen) the rail and the fixed points; to choose a fixed line segment to model the bar KL; to identify a point to be used to direct the motion (it may be either a free point, if any, or a point bound to the rail);
(b) to draw the digital copy of the instrument around the chosen elements; this has to be done following the software logic on the one hand, and the geometrical properties of elements, on the other hand. So, for instance, when one knows point A and point K, to find point I, one needs to construct a right angle, which can be done using the features of semicircles…

Third part: back in the world of the dynamic instrument.

(a) to interpret the digital copy of the instrument; to drag it;
(b) to verify whether the fixed rulers maintain fixed length; to check whether the arch is drawn (see Fig. 13);
(c) to check whether the digital copy of the instrument works in the same way as the concrete one; to analyse limits (e.g. is the same arch traced?) and potentialities (may the fixed line segment KL, a 'parameter' of the construction, be changed?) What happens if this parameter is changed?

Changing the length of some line segments in the digital model has different consequences:

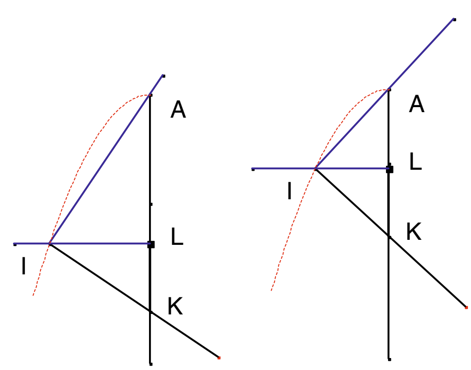

Fig. 13 Two different frames of the same digital instrument

- changing the line segments containing AI and IK changes the length of the arc on the same parabola;
- changing the length of the line segment KL changes the width of the parabola.

A practical application of the modelling process is realized when it is necessary to build a new concrete model: the measure of the wooden board and of the bars may be designed carefully before cutting them, to obtain a well proportioned arc.

4 Concluding remarks

The three examples show clearly how an old tradition, rooted on Klein's views, may be resumed today. In all cases, the reference to the history of mathematics is explicit and brought to the students' knowledge, concrete models and dynamic instruments are available for students' real manipulation and complementarities between cultural aspects and modelling are pursued.

Last, but not least, information and communication technologies are introduced, although in different forms and with different aims:

- In the Cornell project, professional movies show the actual functioning of the precious historical instruments and stereo lithographic files allow 3-D printing of some of the models; moreover, other interactive simulations allow Web exploration of dynamic instruments;
- In the CRICED project, historical books are weaved together with interactive dynamic simulations;
- In the mathematical machines project, dynamic simulations are not only available on the Web, but also are objects of specific teaching and learning activities with prospective mathematics teachers offering a non-trivial context for mathematical modelling.

There is no claim that concrete models and dynamic instruments may be replaced by their digital copies with no loss. Trivially, the digitalization of instruments allows them to become widely available: where there is an access to the Internet one can play with these models interactively. Yet a deep analysis of the changes (if any) in both didactical and cognitive processes when a concrete object is replaced by a digital copy is yet to be performed.

As mentioned earlier, mathematical modelling is not the only element, but an important element of all three research programmes. In this study, we have shown that modelling and application can be paired within an approach that does not neglect, but rather emphasize, the cultural aspects of mathematics, going back to the prominent founders of modern mathematics and taking advantage of the increasingly wider diffusion of information and communication technologies.

Acknowledgments We are very grateful to David W. Henderson for his help with the final version of this manuscript. M.G. Bartolini Bussi was funded by the project PRIN 2007B2M4EK on "Instruments and representations in the teaching and learning of mathematics: theory and practice". D. Taimina was partially funded by National Science Foundation (research grant DUE-0226238) and Institute of Museum and Library Services (research grant LG-30-04-0204-04). M. Isoda was funded by the project JSPS 17300243 on "Developing Mathematics Exhibition for Science Museum".

References

Bartolini Bussi, M. G. (1998). Drawing instruments: Theories and practices from history to didactics. *Documenta mathematica, extra volume ICM98*, pp. 735–746.

Bartolini Bussi, M. G. (2001). The geometry of drawing instruments: Arguments for a didactical use of real and virtual copies. *Cubo, 3*(2), 27–54.

Bartolini Bussi, M. G., & Pergola, M. (1996). History in the mathematics classroom: Linkages and kinematic geometry. In H. N. Jahanke, N. Knoche, & M. Otte (Eds.), *Geschichte der Mathematik in der Lehre* (pp. 36–67). Goettingen: Vandenhoek & Ruprecht.

Campedelli, L. (1958). I Modelli Geometrici. In Commissione internazionale per lo studio e il miglioramento dell'insegnamento della matematica (Ed.), *Il Materiale per L'insegnamento della Matematica* (pp. 143–172). Firenze: La Nuova Italia.

Castelnuovo, G. (1928). *La Geometria Algebrica e la Scuola Italiana. Atti del Congresso Internazionale dei Matematici* (Vol. I, p. 194). Bologna: Zanichelli.

Castelnuovo, E. (2008). *L'Officina Matematica*. Bari: Edizioni La Meridiana.

Cavalieri, B. (1632). *Lo specchio ustorio (2001)*. Firenze: Giunti.

Clairaut, A. C. (1741). *Élements de Geometrie (2006)*. Éditions Jacques Gabay.

Comenius, I. A. (1657). *Didactica Magna (1986)*. Akal Ediciones.

Fischer, G. (Ed.). (1986). *Mathematical models: From the collections of Universities and Museums*. Braunschweig/Wiesbaden: Friedr. Vieweg & Sohn.

France National Constitution. (1791). http://sourcebook.fsc.edu/history/constitutionof1791.html. Accessed 25 August 2009.

Giacardi, L. (2008). *Bibiligograpy of Castelnuovo, G.* http://www.icmihistory.unito.it/portrait/castelnuovo.php. Accessed 21 November 2008.

Henderson, D., & Taimina, D. (2005a). *Mathematical aspects of the Peaucellier-Lipkin linkage*. http://kmoddl.library.cornell.edu/tutorials/11/. Accessed 21 November 2008.

Henderson, D., & Taimina, D. (2005b). *Experiencing geometry: Euclidean and non-euclidean with history*. Upper Saddle River: Pearson Prentice Hall.

Isoda, M. (2004). The development of mathematics education. In Japan International Cooperation Agency (Ed.), *The history of Japan's educational development* (pp. 159–172). Tokyo: Institute for International Cooperation.

Isoda, M. (2008). Sample videos developed by Isoda, M. and Baldin, Y. Y. in 2008 at UFRJ on HTEM4. http://math-info.criced.tsukuba.ac.jp/software/dbook/dbook_eng/dbook_eng05.html. Accessed 21 November 2008.

Isoda, M., Hara, K., Iijima, Y., Uehara, E., & Watanabe, H. (2006). e-Textbooks: Advancing mathematics teaching and learning into the future mathematics. In Chu, S. C. et al. (Eds.), *Proceedings of the eleventh Asian technology conference in mathematics* (pp. 54–63). VA: ATCM, Inc.

Isoda, M., & Matsuzaki, A. (2003). The roles of mediational means for mathematization: The case of mechanics and graphing tools. *Journal of Japan Society of Science Education, 27*(4), 245–257.

Isoda, M., Suzuki, A., Ohneda, Y., Sakamoto, M., Mizutahi, N., & Kawazaki, N. (2001). LEGO project. *Tsukuba Journal of Educational Study in Mathematics, 20*, 77–92.

Kempe, A. B. (1877). *How to draw a straight line*. London: Macmillan.

Klein, F. (1924–1925). *Elementary mathematics from an advanced standpoints.* (Hedrick, E. R., & Noble, C. A., translated, 1939). NY: Macmillan.

Klein, F., & Riecke, E. (1904). *Neue Beiträge zur Frage des Mathematischen und Physikalischen Unterrichts an den Höheren Schulen*. Leipzig: Teubner.

Koenigs, G. (1897). *Leçons De Cinematique* Paris A. Hermann. http://www.archive.org/stream/leonsdecinma00koenuoft#page/242/mode/2up. Accessed 25 August 2009.

Kuroda, M. (1920). *Geometry for secondary school*. Tokyo: Baifu-kan.

Kuroda, M. (1927). *New issues of mathematics teaching (selected publication of his posthumous manuscripts)*. Tokyo: Baifu-kan.

Le Blanc, M. (n.d.). *Friedrich Froebel: His life and influence on education.* http://www.communityplaythings.co.uk/resources/articles/friedrich-froebel.html.

Maclaurin, C. (1720). *Geometria Organica, Sive Descriptio Linearum Curvarum Universalis*. Londini.

Maschietto, M. (2005). The laboratory of mathematical machines of modena. *Newsletter of the European Mathematical Society, 57*, 34–37.

Monbusyo Approved. (1943). *Mathematics textbooks for secondary school (II)*. Tokyo: TyutoGakkoKyokasyo Pub.

Moon, F. C. (2002). *Franz Reuleaux: Contributions to 19th C. kinematics and theory of machines*. http://techreports.library.cornell.edu:8081/Dienst/UI/1.0/Display/cul.htmm/2002-2?abstract=. Accessed 21 November 2008.

Mueller, W. (2001). Mathematical Wunderkammern. *MAA Monthly, 108*, 785–796.

Palladino, N. (n.d.). http://www.dma.unina.it/~nicla.palladino/catalogo/. Accessed 25 August 2009.

Pan, B., Gay, G. K., Saylor, J., Hembrooke, H. A., & Henderson, D. (2004). Usability, learning, and subjective experience: User evaluation of K-MODDL in an undergraduate class. In H. Chen, M. Christel, & E. Lim (Eds.), *Proceedings of the fourth ACM/IEEE joint conference on digital libraries (JCDL' 04)* (pp. 188–189). New York: ACM.

Parshall, K. H., & Rowe, D. (1991). *The emergence of the American mathematical research community 1876–1900: J. J. Sylvester, Felix Klein, and E. H. Moore*. Providence: American Mathematical Society, London Mathematical Society (Co-publication).

Perry, J. (1913). *Elementary practical mathematics*. London: Macmillan.

Reuleaux, F. (1876). *The kinematics of machinery: Outlines of a theory of machines*. London: Macmillan (reprinted New York: Dover, 1963).

Ruthven, K. (2008). *Mathematical technologies as a vehicle for intuition and experiment: a foundational theme of the ICMI, and a continuing preoccupation*. Paper prepared for the ICMI centennial symposium, Rome 2008. http://www.unige.ch/math/EnsMath/Rome2008/WG4/Papers/RUTHVEN.pdf. Accessed 21 November 2008.

Sanden, H. V. (1914). *Praktishe analysis*. Leipzig und Berlin: Teubner.

Schilling, M. (1911). *Catalog matematischer Modelle für den Höheren Mathematischen Unterricht*. Leipzig: Martin Schilling.

Schooten, F. V. (1646). *De organica Conicarum Sectionum Constructione*. Leyden.

Schubring, G. (1989). Pure and applied mathematics in divergent institutional settings. In D. E. Rowe & J. McCleary (Eds.), *Ideas and reception* (pp. 171–207). Boston: Academic Press.

Shell-Gellasch, A., & Acheson, B. (2007). Geometric string models of descriptive geometry, hands on history. In A. Shell-Gelasch (Ed.), *MAA notes #72* (pp. 49–62). Washington, DC: MAA.

Smith, D. E. (1913). Intuition and experiment in mathematical teaching in the secondary schools. *L'Einseignement Mathématique*, *14*, 507–534.

Taimina, D. (2005a). *Linkages.* http://kmoddl.library.cornell.edu/linkages/. Accessed 21 November 2008.

Taimina, D. (2005b). *How to draw a straight line*. http://kmoddl.library.cornell.edu/tutorials/04/. Accessed 21 November 2008.

Taimina, D., & Henderson, D. (2005). *Reuleaux Triangle, learning module*. http://kmoddl.library.cornell.edu/tutorials/02/. Accessed 21 November 2008.