Abstract of the talk *"Replica Symmetry Breaking in Neural Networks"* Linda Albanese "Università del Salento"

In recent decades, the disruptive rise of Artificial Intelligence solutions has profoundly affected current science. Its repercussions influence the framework of applied disciplines (see e.g. [1, 2]), while at the same time it stimulates a theoretical interest in automated systems (from neuroscience to statistics and physics of complex systems, via engineering and information theory).
As far as the mathematical physics concerns, in the context of complex systems, and thus of all those AI models that want to be modelled through them, a major issue is certainly that of replica symmetry breaking [3], which nowadays is considered the core of the complex behaviour of these systems. Complex systems share a dynamics spread over several timescales, their free energy exhibits a huge amount of disordered valleys, whose number depends on the system size. Moreover, these systems experience "aging", namely if an experiment on these systems takes a week and we perform twice the same experiment on the same system at, for instance, a temporal distance of one year (this "waiting time" must be larger than the typical characteristic scale of the experiment), we will collect different results from the two trials -intuitively because in the waiting time the system may have shifted from one free energy minimum into another (that is why this phenomenon is called "aging"[4]).

Purpose of this presentation, after the introduction of all the main definition we need, will be to describe this phenomenon and a rigorous mathematical approach to solve some models of neural network assuming it.
To do so, we will use Guerra's interpolation [5], which is simpler in computations and mathematically justified in every passage. Moreover, Francesco Guerra has already introduced a way to deal with RSB regimes [6] in spin glasses, known as *ziqqurat* ansatz. Therefore, in this presentation we will see how to import these techniques in neural networks and, in particular, to Hopfield model, the archetypal of them.

The talk is based on joint work with E. Agliari (Sapienza Università di Roma), A. Barra (Università del Salento) and G. Ottaviani [7].

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Abstract (M. Alessandro's PhD thesis)

In this thesis we treat two topics: the construction of minimal complex surfaces of general type with $p_g = q = 2, 3$ and an extension of Schur's concept of a representation group for projective representations to the setting of semi-projective representations. These are the contents of the two articles [AC22] and [AGK23], which are two joint works: the former with Fabrizio Catanese, the latter with Christian Gleissner and Julia Kotonski.

The first part of the thesis is devoted to the treatment of the construction method for minimal surfaces of general type with $p_g = q$ developed together with Fabrizio Catanese in [AC22].

We give first a construction of minimal surfaces of general type with $p_g = q = 2$, $K^2 = 5$ and Albanese map of degree 3, describing a unirational irreducible connected component of the Gieseker moduli space, which we show to be the only one with these invariants fulfilling a mild technical assumption (called in the thesis *Gorenstein Assumption*) and whose general element S has Albanese surface $\mathrm{Alb}(S)$ containing no elliptic curve. We call it the component of *CHPP surfaces*, since it contains the family constructed by Chen and Hacon in [CH06], and coincides with the one constructed by Penegini and Polizzi in [PePo13a].

Similarly, we construct a unirational irreducible connected component of the moduli space of minimal surfaces of general type with $p_g = q = 2$, $K^2 = 6$ and Albanese map of degree 4, which we call the component of *PP4 surfaces* since it coincides with the irreducible one constructed by Penegini and Polizzi in [PePo14].

Furthermore, we answer a question posed by Chen and Hacon [CH06] by constructing three families of surfaces with $p_g = q$ whose Tschirnhaus module has a kernel realization with quotient a nontrivial homogeneous bundle. Two families have $p_g = q = 3$ (one of them is just a potential example since a computer script showing the existence is still missing), while the third one is a new family of surfaces with $p_g = q = 2$, $K^2 = 6$ and Albanese map of degree 3. The latter, whose existence is showed in [CS22], yields a new irreducible component of the Gieseker moduli space, which we call the component of *AC3 surfaces*. This is the first known component with these invariants, and moreover we show that it is unirational.

We point out that we provide explicit and global equations for all the five families of surfaces we mentioned above.

Finally, in the second and last part of the thesis we treat the content of the joint work [AGK23] with Christian Gleissner and Julia Kotonski.

Here we study *semi-projective representations*, i.e., homomorphisms of finite groups to the group of semi-projective transformations of finite dimensional vector spaces over an arbitrary field K. The main tool we use is *group cohomology*, more precisely explicit computations involving cocycles.

As our main result, we extend Schur's concept of *projective representation groups* [Sch04] to the semi-projective case under the assumption that K is algebraically closed.

Furthermore, a computer algorithm is given: it produces, for a given finite group, all *twisted representation groups* under trivial or conjugation actions on the field of complex numbers.

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UNIVERSITY OF PALERMO

Abstract

Department of Mathematics and Computer Sciences

Doctor of Philosophy

Enhancing Computational Fluid Dynamics with Artificial Intelligence: an AI-based Smoothed Particle Hydrodynamics (SPH) Emulator for Lava Flow Modeling

by Eleonora AMATO

The combination of Computational Fluid Dynamics (CFD) and Artificial Intelligence (AI) expands the scope of fluid modeling, providing high fidelity and fast simulations. A variety of Eulerian CFD methods integrated with AI has been already successfully presented (*e.g.*, for weather forecasting); on the other hand, the combination of AI and Lagrangian methods remains less consolidated. Smoothed Particle Hydrodynamics (SPH) is a Lagrangian mesh-less CFD numerical method, highly reliable for the simulation of complex fluids. Nevertheless, SPH models exhibit limitations in high-resolution real-time simulations of physical phenomena, due to the high computational costs involved. Specifically, SPH simulations of lava flows are well representative of the difficulties in modeling highly complex fluids. Lava is a fluid with a high physical complexity, generating viscous flows, dependent on temperature and rheology, and it may have significant impacts on the surrounding environment. Thus, it is important to monitor lava flows with accurate and timely forecasting of their spatio-temporal evolution. Here, I present an emulator derived from CFD physics-based models, in which AI algorithms join the equation-based mathematical representation of physics, to solve fluid dynamics problems in shorter times. I developed an AI-based emulator for SPH method, in which the conservation of momentum equation is substituted by an Artificial Neural Network (ANN), which learns from SPH simulations. The ANN is trained to estimate SPH particles interaction forces exploiting as input the state of the particles (position, velocity, density). I verified the reliability of the AI-based emulator to remain as faithful as possible to the SPH reference model. Applications to different kind of fluids are shown, starting from an inviscid fluid, up to the study of a viscous fluid with a thermal component, to finally move towards the description of a lava flow evolution, exploiting the potential of the combined use of numerical and AI models. Simulations and emulations have been compared for each step, reaching a high degree of fidelity, and demonstrating the generalizability of the AI-based emulator, tested over problems with varying levels of complexity, and its robustness to different spatial resolutions.

On the gradient rearrangement of a function: the BV case

Vincenzo Amato

Abstract

Symmetrization techniques are valuable tools in the study of many mathematical problems, for instance in the analysis of PDEs and in many shape optimization problems. This presentation focuses on the gradient symmetrization of a function with bounded variation, not necessarily vanishing at the boundary. The latter is an unusual feature for this type of construction, allowing a wider range of applications.
Applications include classic Saint-Venant inequality for the p-Laplacian with Robin boundary conditions and similar inequalities for functionals related to the optimal insulation of a given domain. These applications suggest that future developments are certainly worth exploring.

The results presented here were obtained in joint work with Andrea Gentile, Carlo Nitsch and Cristina Trombetti.

Well-posedness and stability of slightly compressible Boussinesq's flow in Darcy–Bénard problem

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Thermal convection phenomena in Newtonian fluid-saturated horizontal porous layers heated from below, are in essence physically motivated by buoyant force induced from density variations due to the presence of a thermal gradient across the layer. Indeed, in non-isothermal processes variations in temperature generate variations in the fluid's properties. In particular, concerning the fluid density, an analysis including the full effect of this variation is so complicated that some approximations become essential. The vast majority of the studies concerning the stability of basic steady state motions, in clear fluids as well as in fluid-saturated porous media, are addressed under a celebrated hypothesis, known as the Boussinesq, or Oberbeck-Boussinesq (OB) approximation. However, in [1] Müller proved that as long as one assume as incompressible a medium whose constitutive equations do not depend on pressure p but only on temperature T, then only a *constant density function* wouldn't be in contradiction with the Gibbs law. This conclusion is in disagreement with empirical observations, according to which fluids expand when heated, and the theoretical assumptions such as the OB approximation. For this reason the above conclusion was called *Müller paradox*. In order to fix this contradiction, a less restrictive model of incompressibility was proposed by Gouin and Ruggeri [2]. In particular, enforcing essential thermodynamic conditions, namely entropy principle and thermodynamic stability, the class of *extended-quasi-thermal-incompressible fluids* (EQTI) was introduced. For this class of fluids, the following constitutive law for the fluid density ρ is considered:

$$\rho(p,T) = \rho_0[1 - \alpha(T - T_0) + \beta(p - p_0)], \tag{1}$$

where ρ_0 is the reference density in correspondence of a temperature T_0 and pressure p_0, α is the thermal expansion coefficient and β is the compressibility factor.
The objective of the present talk is to show the well-posedness of the mathematical model describing the flow of a EQTI fluid saturating a Darcy porous medium described by the following system

$$\begin{cases} \dfrac{\mu}{K}\mathbf{v} = -\nabla p - \rho_0[1 - \alpha(T - T_0) + \beta(p - p_0)]g\mathbf{k}, \\ \nabla \cdot \mathbf{v} = 0, \\ \rho c_V\Big(\dfrac{\partial T}{\partial t} + \mathbf{v}\cdot\nabla T\Big) = \chi\Delta T, \end{cases} \tag{2}$$

to which we associated isothermal boundary conditions. Moreover, the critical Rayleigh-Darcy number for the onset of convection is determined, via instability analysis, as a function of a dimensionless parameter $\widehat{\beta}$, proportional to the compressibility factor β, proving that $\widehat{\beta}$ enhances the onset of convective motions [4].

The talk is based on a joint work with F. Capone, R. De Luca and G. Massa.

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Researching the Teaching and Learning of Physics in Formal and Informal Contexts: A Study from Kindergarten to University

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The process of modelling physical phenomena and acquiring mathematical formalisation can be challenging in physics education at all levels. Therefore, it is recommended that interpretative and mathematical models be developed gradually from an early age, starting from kindergarten. Even young children possess innate abilities of abstraction, discerning discrete quantities, and spatial thinking (Dehaene,2009;2011), making it possible to introduce them to the basics of modelling physical phenomena. A phenomenological approach to the modeling process (Lehrer & Schauble, 2015) is the cornerstone of my research. To understand how people model phenomena, I have leveraged the resonance model as a means of inquiry (Guidoni et al. 2005a). The cognitive resonance model is a valuable tool for observation and didactic design with the possibility of linking phenomenological experiments to neuroscientific, epistemological, and psychological learning theories. Our interventions in schools refer to a revisited approach of the Design Experiment (e.g., Cobb et al. 2003; diSessa & Cobb, 2004). We design and act in teacher training courses and work in classes interacting with students and teachers. The activities are laboratory-based, and students are divided into small groups to analyse and model phenomena by interacting with peers, teachers, and researchers. Using mature and new technologies, students are asked to represent relationships between the quantities that characterise a phenomenon and argue the hypotheses that make an interpretative and mathematical model plausible. The findings from the evaluations shared with teachers show that people of different contexts can develop considerable abstraction skills in handling models if the teaching activity has been well designed and managed. This result refers to such an early meta-modelling activity, which is not generally foreseen in science education standards (NGSS,2013). The results emerge from action-research activities implemented in formal and informal contexts and analysed qualitatively using a content analysis approach (Mayring, 2014, 2020).

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Fueter and Almansi theorems for slice regular functions of several quaternionic variables

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We broaden some definitions and give new results about the theory of slice functions of several quaternionic variables. We introduce the notions of partial spherical value and derivative for functions of several variables that extend those of one variable, recovering some of their properties and discovering new ones. This leads to a generalization of Fueter's theorem for slice regular functions of several quaternionic variables. Furthermore, partial spherical derivatives can be use to obtain different Almansi decompositions for slice functions of several variables. The components of each decomposition, defined explicitly through partial spherical derivatives, exhibit desirable properties, such as harmonicity and circularity. As consequences of these decompositions, we give another proof of Fueter's theorem in $\mathbb{H}^n$, establish the biharmonicity of slice regular functions in every variable and, time permitting, derive some integral formulas for them.

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II meeting UMI for Doctoral Students - Talk proposal

Lucrezia Bottegoni, SSD: MAT/02

Title. **On semiseparability, semifunctors and conditions up to retracts**

Abstract. The notion of separability is crucial in several topics in Algebra, Number Theory and Algebraic Geometry, for instance in classical Galois theory, ramification theory, Azumaya algebras, and Hochschild cohomology. Separable functors were introduced in [9] in order to reinterpret categorically the theory of separable field extensions, and of separable algebras. Several results and applications of separable functors are illustrated e.g. in [6]. Explicitly, a functor $F : \mathcal{C} \to \mathcal{D}$ is said to be *separable* if the associated natural transformation $\mathcal{F}_{X,Y} : \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}}(FX, FY)$, mapping f into Ff, has a left inverse, i.e. there is a natural transformation $\mathcal{P}_{X,Y} : \mathrm{Hom}_{\mathcal{D}}(FX, FY) \to \mathrm{Hom}_{\mathcal{C}}(X, Y)$ such that $\mathcal{P}_{X,Y} \circ \mathcal{F}_{X,Y} = \mathrm{Id}_{\mathrm{Hom}_{\mathcal{C}}(X,Y)}$, for all X and Y in $\mathcal{C}$. A dual version of this property arises by requiring that $\mathcal{F}$ has a right inverse and yields the notion of *naturally full* functor, as defined in [3]. In [1] we have defined a functor $F : \mathcal{C} \to \mathcal{D}$ to be *semiseparable* if $\mathcal{F}_{X,Y}$ is a regular natural transformation - an analogue of von Neumann regular element - i.e., if $\mathcal{F}_{X,Y}$ admits a natural transformation $\mathcal{P}_{X,Y}$ such that $\mathcal{F}_{X,Y} \circ \mathcal{P}_{X,Y} \circ \mathcal{F}_{X,Y} = \mathcal{F}_{X,Y}$, for all X and Y in $\mathcal{C}$. It is known that a separable functor is faithful and that a naturally full functor is full: the reverse implications hold by adding the assumption of semiseparability. Through the coidentifier category construction we have shown that any semiseparable functor admits a canonical factorization as a naturally full functor followed by a separable functor. A central result for separable functors is the so-called Rafael Theorem which provides a characterization of separability for the left (resp. right) adjoint in terms of splitting properties of the (co)unit. In the same spirit, we have proved that a functor that has a right (resp. left) adjoint is semiseparable if and only if the (co)unit of the adjunction is regular as a natural transformation. We have then applied the results to functors traditionally attached to ring homomorphisms, coalgebra maps, corings and bimodules. In this talk, after presenting the main properties of semiseparable functors, we discuss semiseparability with respect to functors whose idempotent completion admits a fully faithful left (right) adjoint, that we have called *coreflections up to retracts* (*reflections up to retracts*, respectively) in [2]. One can also consider a stronger notion, that we have called *bireflection up to retracts*, involving both left and right adjoint. We have proved that a right (left) adjoint functor is semiseparable if, and only if, the associated (co)monad is separable and the (co)comparison functor is a bireflection up to retracts, extending a characterization pursued by X.-W. Chen in the separable case [7]. The notions of semifunctor, defined as a functor that does not necessarily preserve identities, and semiadjunction (due to S. Hayashi [8]) have been helpful in this research. As a consequence, given an adjunction, the semiseparability of the right adjoint provides an equivalence after idempotent completion between the Kleisli category of free modules over the associated monad and the Eilenberg-Moore category of modules over that monad. Time permitting, we describe an application of these results in the context of pre-triangulated categories: we provide conditions for the Eilenberg-Moore category of modules to inherit the pre-triangulation from the base category, obtaining a semi-analogue of a result shown by P. Balmer for separable functors [4]. Finally, we present the notion of *semifullness* for semifunctors, introduced and investigated in [5].

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A Constructive Picture of Noetherian Conditions and Well Quasi-Orders

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Constructive mathematics is commonly understood as mathematics done without the Law of Excluded Middle, that is without using $\varphi \vee \neg\varphi$ holds for any formula φ. This apparently limiting condition allows for deeper insights into mathematics especially from a computational perspective.

In [1] we analyse from a constructive point of view the main definitions for well quasi-orders, wqo, we have found in the literature, together with the related concepts of a Noetherian ring or module. Despite being all equivalent to each other in the classical setting, they differ in computational content, as has already been observed in reverse mathematics [2,4].

Fifty years after the discovery of a constructively viable concept of Noetherianity by Richman [6] and Seidenberg [7], we carry out a joint analysis of Noetherianity and wqo, in such a way that the rich literature of the former can be made useful for to the latter. We thus aim at a more comprehensive picture of partial and quasi-order properties in the spirit of constructive reverse mathematics [3,5]. Among other things we find a countermodel for a hybrid Noetherian condition whose constructive status was open.

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Unrefinable Partitions into Distinct Parts

Lorenzo Campioni "Università dell'Aquila"

A partition into distinct parts is refinable if one of its parts a can be replaced by two different integers which do not belong to the partition and whose sum is a, otherwise is unrefinable. For example, the partition $(1, 2, 5, 7)$ is refinable because we can write 7 as $3 + 4$, while the partition $(1, 2, 3, 6, 7, 11)$ is unrefinable.
Clearly, unrefinability is a non-trivial limitation on the distribution of the parts. For this reason, after the creation of an algorithm that recognises and enumerates this kind of partitions, the focus was on the size of the largest element, for which it was found an upper bound, and on the number of partitions which reach the bound, that we call maximal.
First of all, the study starts with the easier problem of counting maximal unrefinable partitions of the triangular number $T_n = \frac{n(n+1)}{2}$. We prove that, if n is even, then there exists only one maximal unrefinable partition of T_n, while if n is odd we show a one-to-one correspondence between the maximal unrefinable partitions of T_n and the partitions in distinct parts of k, where $n = 2k - 1$.
After that we generalize the case of any integer $T_{n,d} = T_n - d$ and we obtain again a complete classification of maximal unrefinable partitions and we show, again, that this is related to suitable partitions into distinct parts, depending on the distance from the successive triangular number.
In the last part of the PhD we found some relations between unrefinable partitions and numerical semigroups , subsets $S \subset \mathbb{N}$ such that the sum of two elements in S is in S and the complementary of S is a finite subset of $\mathbb{N}$.

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Parallel Pathways: Generative Models and Onsager Reciprocal Relations through the lens of Nonequilibrium Statistical Physics

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Abstract

At first glance, the two topics addressed in each section of the present PhD thesis may seem orthogonal: indeed, generative models and linear response theory appear to have little overlap. However, many interesting research topic are linked by common themes after a deep analysis. In the present case, the central narrative thread revolves around Nonequilibrium Statistical Physics. The first fundamental tool from that field that has been used in the present work is Jarzynski identity (C. Jarzynski, 1997); it provides a connection between microscale and macroscale, relating microscopic work along trajectories and free energy, respectively. On the other hand, Onsager reciprocal relations (ORR) represent a milestone in that area (L. Onsager, 1931): they serve as a bridge between a microscopic property (time reversal symmetry) and a macroscopic one (response tensors).

In the first section, we show how *recent* theoretical results can be very instrumental in state-of-the-art applications [1]; generative model represent a substantial research challenge since they are already used in everyday life, even if we are far from having a complete theoretical picture about them. In a nutshell, we propose a novel training algorithm for energy-based models (EBMs), which is a class of diffusion generative models strongly inspired by Statistical Physics, namely by Boltzmann-Gibbs ensemble; in light of this relation, a key strength of EBMs compared to other models is their interpretability. Standard procedures, such as those based on Constrastive Divergence, heavily relies on approximations of the real loss objective already in an ideal setup. Because of that, the practical implementation of such methods usually requires a lot of empirical tricks, often not theoretically justified. In contrast, our proposal is exact; furthermore, no extra bias is introduced by discretization in time and the algorithm provides for free additional information on the trained EBM (i.e. the normalization constant of the trained probabilistic model). Our contribution is based on Jarzynski identity in continuous time and Annealed Importance Sampling in discrete time. As a side note, the first section covers most of the PhD thesis: we retain important to contextualize our result in a monography on the links between EBMs and Statistical Physics, in order to provide an useful review of the state of the art in this area.

In the second section, we show that *established* theoretical findings can be still object of refinements [2][3]. ORRs basically provide information on the structure response tensors, reducing the degrees of freedom of such objects; the main request for such relations to hold is canonical time reversal symmetry, i.e. the invariance of the equations of motion under the inversion of velocities. Our work demonstrates how we can relax this condition by expanding upon the definition of time reversal symmetry. This expansion enables us to prove that the set of symmetries leading to time reversal invariance is broader. In particular, we explicit a taxonomy of generalized time reversal operations and a set of compatibility conditions between those transformation and a generic external magnetic field. The experimental validity of ORRs has been proven in many contexts where canonical time reversal seems to not hold. Thus, our result contributes to explain some of these examples.

Featured papers

[1] D. Carbone, M. Hua, S. Coste, and E. Vanden-Eijnden, *Efficient Training of Energy-Based Models Using Jarzynski Equality*, Proceedings of 37th Conference on Neural Information Processing Systems (NeurIPS), 2023

[2] D. Carbone, P. De Gregorio, and L. Rondoni, *Time reversal symmetry for classical, non-relativistic quantum and spin systems in presence of magnetic fields* , Annals of Physics 441, 2022

[3] D. Carbone and L. Rondoni. *Necessary and Sufficient Conditions for Time Reversal Symmetry in Presence of Magnetic Fields.* Symmetry 12.8, 2020

Sorting with a popstack

Lapo Cioni * Luca Ferrari †

Abstract

The study of combinatorial properties of sorting algorithms was introduced by Knuth [1], which discovered that the permutations that could be sorted using a special container, called *stack*, are those satisfying a specific restriction. Subsequently, West [2] rewrote the result using the concept of *pattern* of a permutation, while also finding other properties of the algorithm.

Informally, a stack is a container in which elements can be stacked on top of each other, so that we can only recover them in inverse order with respect to the one in which they were inserted. In this work we consider a *popstack*, which stores elements in the same way as a stack, but they can only be retrieved together, by emptying the popstack. We use a popstack with a bypass to sort a permutation. That is, we have a permutation $\pi = \pi_1 \dots \pi_n$ and we want to output $12 \dots n$. To do this, we consider the elements of the permutation one at a time, and choose wether to send them directly to the output, insert them in the popstack, or empty the popstack. Therefore, these are the allowed operations:

- (*insert*): insert the current element of the input in the popstack, on top of all the other elements;
- (*pop*): remove all the elements in the popstack, in order from top to bottom, sending them into the output.
- (*bypass*): output the current element of the input.

We describe a sorting algorithm, which we call `Popstacksort`, and prove that it is optimal, in the sense that it sorts every sortable permutation. We then investigate the properties of `Popstacksort`. Specifically, we descrive the preimages under the algorithm of any single permutation, and the preimages of any principal permutation class, as described in Table 1.

Finally, we use the previous results to describe the permutations that can be sorted composing `Popstacksort` with other well-known sorting algorithms, such as `Bubblesort`, `Queuesort` and `Stacksort`.

ρ	$\mathbf{PS}^{-1}(\mathrm{Av}(\rho))$	Notes
$\emptyset$	$\mathrm{Av}(\emptyset)$	
1	$\mathrm{Av}(1)$	
12	$\mathrm{Av}(12, 21)$	
$n\alpha$	$\mathrm{Av}(n(n+1)\alpha, (n+2)n\tau)$	for every $\tau \in n+1 \sqcup \alpha$, $\tau \neq n\alpha$
$(n-1)\alpha n$	$\mathrm{Av}((n-1)n\alpha, (n+1)(n-1)\tau)$	for every $\tau \in n \sqcup \alpha$, $\tau \neq n\alpha$
Anything else	Not a class	

Table 1: The preimages under `Popstacksort` of all principal classes.

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Congested optimal transport in the Heisenberg Group

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Abstract

Scope of this thesis is to study the congested optimal transport problem (COTP) in the Heisenberg Group, equipped with a sub-Riemannian metric.

In the Euclidean setting, COTP has been introduced first as a discrete traffic problem on networks (see [5]); afterwards, in [2] Carlier et al. expressed it as a Monge–Kantorovich-type problem, which takes into account congestion effects.

The Heisenberg group $\mathbb{H}^n$ is the simplest sub-Riemannian manifold: it is characterized by the choice of both $2n$ vector fields and a metric on the sub-bundle of the tangent bundle generated by them. Since $2n$ is strictly smaller than the dimension of the space, the metric is degenerate at every point. All the intrinsic objects of the space are defined in terms of the given vector fields; in particular displacement takes place only along the integral curves of them, called horizontal.

In the COTP, the distributions of residents and services in a bounded region $\Omega \subseteq \mathbb{H}^n$ are expressed via two probability measures μ, ν, supported in Ω. In [4] we modelled the way in which commuters choose their paths (to get some services) through probability measures over the set of horizontal curves. As in [2], we obtain the existence of equilibrium configurations, as solutions of a minimization problem over a suitable set of probability measures over horizontal curves. To obtain the previous result the main difficulty was represented by the geometry of the space.

In the spirit of [1], in [3] we show that the aforementioned problem has an equivalent formulation, as a minimization problem over a set of horizontal vector fields with prescribed horizontal divergence. To do this, we pass through a sequence of Riemannian manifolds that approximates $\mathbb{H}^n$ in the Gromov-Hausdorff sense. The previous problem admits a dual formulation, which leads to degenerate p-Laciacian type PDEs: the main obstacle in studying these equations is the non-commutativity of the given vector fields.

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Opinion formation and flocking models with attractive-repulsive interaction

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Abstract

In these last years, multiagent systems have been studied by several authors, due to their wide application to many scientific disciplines, such as biology, economics, control theory, and engineering. Among them, there are the Hegselmann-Krause opinion formation model and its second-order version, the Cucker-Smale model, introduced for the description of flocking phenomena. Typically, for the solutions of such models, the convergence to consensus, in the case of the Hegselmann-Krause model, and the exhibition of asymptotic flocking, in the case of the Cucker-Smale model, are investigated.

Here, we analyze Hegselmann-Krause and Cucker-Smale type models in presence of an attractive-repulsive interaction. To be precise, we examine the situation in which the individuals involved in an opinion formation process or in a flocking phenomenon attract each other in certain time intervals and repeal each other in certain other time intervals. Of course, the fact that the agents of the system are able to repeal each other for certain times represents an obstacle to the achievement of the asymptotic consensus and flocking.

We provide conditions for which the solutions of the Hegselmann-Krause model converge to consensus and the solutions of the Cucker-Smale model exhibit asymptotic flocking despite the presence of negative interaction coefficients. In order to get the asymptotic consensus or the asymptotic flocking, one has to compensate the behaviour of the solutions to the considered models in the bad intervals, i.e. the intervals in which the agents repeal each other, with the good behaviour of the solutions of the system in the intervals in which the ifluence among the agents is positive. Also, some restrictions on the lengths of the intervals of negative interaction have to be required.

Under quite general assumptions, we establish the convergence to consensus for the Hegselmann-Krause model with attractive-repulsive interaction and the exhibition of asymptotic flocking for the Cucker-Smale model with attractive-repulsive interaction. Under some additional conditions, we prove that the solutions of the Hegselmann-Krause model in presence of positive-negative interaction converge exponentially to consensus.

Joint work with C. Pignotti.

STRICHARTZ ESTIMATES FOR THE DIRAC EQUATION ON COMPACT MANIFOLDS WITHOUT BOUNDARY

ELENA DANESI

The Dirac equation on $\mathbb{R}^n$ can be listed within the class of dispersive equations, together with, e.g., the wave and Klein-Gordon equations. In the years a lot of tools have been developed in order to quantify the dispersion of a system. Among these one finds the Strichartz estimates, that are a priori estimates of the solutions in mixed Lebesgue spaces. For the flat case $\mathbb{R}^n$ they are known, as they are derived from the ones that hold for the wave and Klein-Gordon equations. However, when passing to a curved spacetime domain, very few results are present in the literature. In this talk I will firstly introduce the Dirac equation on curved domains. Then, I will discuss the validity of this kind of estimates in the case of Dirac equations on compact Riemannian manifolds without boundary. This is based on a joint work with Federico Cacciafesta (Università di Padova) and Long Meng (CERMICS-École des ponts ParisTech).

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Absence of Lavrentiev Phenomenon for Functionals with (p, q)-growth

Filomena De Filippis

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Abstract

We study the absence of Lavrentiev phenomenon for functionals of the following type

$$\mathcal{F}(u) := \int_\Omega f(x, Du(x))\, dx,$$

where $\Omega \subset \mathbb{R}^n$ and $x \mapsto \frac{\partial f}{\partial z}(x, z)$ is α-Hölder continuous. Moreover, the density f is convex and satisfies the (p, q)-growth condition

$$|z|^p \leqslant f(x, z) \leqslant L(1 + |z|^q),$$

with

$$1 < p < q < p + \frac{p\alpha}{n}. \tag{1}$$

For the model density represented by the double phase functional

$$f(x, z) := |z|^p + a(x)|z|^q,$$

we can do better, we can replace the relation (1) with

$$1 < p < q < p + \varkappa,$$

where $\varkappa \in (0, +\infty)$, provided

$$a(x) \leqslant C\left[a(y) + |x - y|^\varkappa\right].$$

The talk is based on the results contained in [1–4].

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A Hahn-like construction for mixed characteristic valued fields

Anna De Mase *

In their paper ([2]), Ax and Kochen give a complete axiomatization of the theory of the class of formally p-adic fields, by having the following properties:

- the value group is a $\mathbb{Z}$-group;
- the residue field is $\mathbb{F}_p$;
- $v(p)$ is the minimal positive element of the value group.

Thus, a valued field K is formally p-adic if and only if it is elementarily equivalent to $\mathbb{Q}_p$. Moreover, assuming the Continuum Hypothesis, they give a characterization of ω-pseudo complete formally p-adic fields with a fixed value group G of cardinality $\aleph_1$, using a Hahn-like construction over $\mathbb{Q}_p$ that preserves the residue field. In this construction, the elements of the field are "twisted" power series over $\mathbb{Q}_p$, i.e. power series whose product is defined by having a power of p as an extra factor. In this talk, we generalize this construction to the more general setting of mixed characteristic henselian valued fields with a cross-section, and we show that an analogous characterization holds for ω-pseudo complete valued fields with finite ramification and valued in a $\mathbb{Z}$-group G.

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A Twin correspondence for prescribed mean curvature graphs in Killing submersions

Andrea Del Prete

Abstract

A Riemannian (resp. Lorentzian) Killing submersion is a Riemannian submersion from a Riemannian (resp. Lorentzian) 3-manifold E onto a Riemannian surface M, both connected and orientable, whose fibers are the integral curves of a non-vanishing (temporal) Killing vector field. In this setting we give a suitable definition of the graph of a smooth function defined over an open subset of M, and we prove a generalized Calabi-type duality between (spacelike) graphs of prescribed mean curvature in Riemannian and Lorentzian Killing submersions. Finally we use this result to prove existence and non-existence results for entire graphs. It is based on a joint work with H. Lee and J. M. Manzano [1].

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ANTI-QUASI-SASAKIAN MANIFOLDS: GEOMETRIC PROPERTIES AND OBSTRUCTIONS

DARIO DI PINTO

Generally considered as the odd dimensional counterpart of almost Hermitian manifolds, almost contact metric manifolds are Riemannian manifolds endowed with a special geometric structure given by a hyperplane distribution $\mathcal{D}$, an almost Hermitian structure J on it, and an orthogonal unitary vector field ξ. Among such manifolds several remarkable classes can be distinguished, the most studied ones being contact metric, Sasakian, cokähler and quasi-Sasakian manifolds.
Recently, the new class of *anti-quasi-Sasakian manifolds* (aqS manifolds for short) has been introduced in [2]. The characteristic feature of aqS manifolds is to be non-normal almost contact metric manifolds, locally fibering along the 1-dimensional foliation generated by ξ onto Kähler manifolds endowed with a closed 2-form of type $(2, 0)$. Hyperkähler manifolds are well known examples of Kähler manifolds with a nondegenerate closed $(2, 0)$-form.
In the present talk, after a brief overview of almost contact metric geometry, I will introduce the new class of anti-quasi-Sasakian manifolds, providing some meaningful examples, including compact nilmanifolds, $\mathbb{S}^1$-bundles and manifolds admitting a $Sp(n) \times 1$-reduction of the structural group of the frame bundle. Then, I will discuss some geometric obstructions to the existence of aqS structures, mainly related to curvature and topological properties. In particular, I will focus on aqS structures of maximal rank on compact homogeneous Riemannian manifolds and on nilpotent Lie groups: in the first case we obtain a non-existence result, while in the latter case we provide a complete classification.

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LONG TIME BEHAVIOR AND STABILITY OF SURFACE DIFFUSION FLOW

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In mathematics a geometric flow is a motion in time of some geometric object or structure, usually driven by a system of partial differential equations. Such geometric evolution equations have been applied to a variety of topological, analytical and physical problems, giving in some cases very fruitful results. We are actually interested in studying the *surface diffusion flow* of smooth hypersurfaces in an n–dimensional flat torus. This flow was first proposed by Mullins in [4] to model thermal growth in material sciences. It is concerned with the physically relevant case of the motion of surfaces in the three–dimensional space describing the evolution in time of interfaces between solid phases of a system, driven by the surface diffusion of atoms under the action of a chemical potential.
According to this flow, smooth hypersurfaces move with the outer normal velocity given by the Laplacian of their mean curvature. Moreover, it can be seen as the H^{-1}–gradient flow of the Area functional and a key geometric property of this flow is the preservation of the volume enclosed by the hypersurfaces during the evolution.
A first local–in–time existence (and uniqueness) theorem was shown by Escher, Mayer and Simonett in [3], then a long–time existence result, in dimension three, was presented in a paper by Acerbi, Fusco, Julin and Morini in [1] (extending previous results for spheres of Escher, Mayer and Simonett [3], Wheeler [5, 6] and Elliott and Garcke [2]). Even if the three–dimensional case is the most relevant from the physical point of view, in a recent work we mainly focus our attention on the generalization of these results to arbitrary dimension. In particular, we show that if the initial set is sufficiently "close" to a *stable critical* set for the Area functional, under a volume constraint, then the flow actually exists for all times and asymptotically converges *in a suitable sense* to a "translation"of the critical set.

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MONGE SOLUTIONS FOR DISCONTINUOUS HAMILTON-JACOBI EQUATIONS IN CARNOT GROUPS

FARES ESSEBEI

In the seminar I would present the Monge solutions to stationary Hamilton–Jacobi equations associated to discontinuous Hamiltonians, in the framework of Carnot groups. The sub-Riemannian Hamilton–Jacobi equation is the form

$$H(x, Xu) = 0 \tag{0.1}$$

where Ω is a subdomain of a *Carnot group* $\mathbb{G}$ of rank m, Xu is the horizontal gradient associated to $\mathbb{G}$ and the Hamiltonian $H : \Omega \times \mathbb{R}^m \longrightarrow \mathbb{R}$ satisfies the following structural assumptions (H):

(H_1) $H : \Omega \times \mathbb{R}^m \to \mathbb{R}$ is Borel measurable;

(H_2) The set

$$Z(x) := \{p \in \mathbb{R}^m \,:\, H(x, p) \leqslant 0\}$$

is closed, convex and $\partial Z(x) = \{p \in \mathbb{R}^m \,:\, H(x, p) = 0\}$ for any $x \in \Omega$;

(H_3) There exist $\alpha > 1$ such that

$$\hat{B}_{\frac{1}{\alpha}}(0) \subset Z(x) \subset \hat{B}_\alpha(0)$$

for any $x \in \Omega$, where $\hat{B}_\alpha(0)$ is Euclidean ball of radius α centered at the origin in $\mathbb{R}^m$.

Following my research article [2] and inspired by [1, 3], we prove the equivalence between Monge and viscosity solutions in the continuous setting. After that, we prove existence and uniqueness for the Dirichlet problem, together with a comparison principle and a stability result.

This is a joint work with Gianmarco Giovannardi and Simone Verzellesi.

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Keywords. Hamilton-Jacobi equations, Carnot groups, Monge solutions, discontinuous Hamiltonians.
MSC. 35F21, 35R03.

MULTIGRADED SYZYGIES OF MONOMIAL IDEALS

ANTONINO FICARRA

The general motif of the dissertation is the study of homological and algebraic properties of homogeneous ideals, with a focus on monomial ideals and ideals with linear powers.

In the paper [3], we introduced the family of *vector–spread strongly stable ideals*, as a vast generalization of (strongly) stable ideals, whose various extensions are considered by many researchers. We compute the Koszul cycles of these ideals and therefore we construct the minimal free resolution. As a consequence, we could compute the graded Betti numbers. We extend the classical Algebraic Shifting theory developed by Kalai and others in this general frame. This theory was recently used to generalize Macaulay theorem and the Bigatti–Hulett theorem in the context of vector–spread ideals.

Homological shift ideals were deeply studied in [1, 2, 4, 6]. Let I be a monomial ideal of a polynomial ring. The ith *homological shift ideal* of I is the monomial ideal $\mathrm{HS}_i(I) = (\mathbf{x}^{\mathbf{a}} : \beta_{i,\mathbf{a}}(I) \neq 0)$. Notice that $\mathrm{HS}_0(I) = I$. A natural question arises. Which properties are enjoyed by all $\mathrm{HS}_i(I)$? A longstanding conjecture, proposed in 2012 by Bandari, Bayati and Herzog, says that all $\mathrm{HS}_i(I)$ are polymatroidal, if I is polymatroidal. We settled this conjecture for $j = 1$ [4], and in degree two [6]. The homological shift ideals of edge and cover ideals of finite simple graphs were studied in [1, 2, 6]. In particular, we conjectured that $\mathrm{HS}_j(J(G)^k)$ has linear quotients for all j and k, if $J(G)$ is the cover ideal of a Cohen-Macaulay very well-covered graph. This conjecture was settled for $k = 1$, and if G is either bipartite or a whisker graph.

The last topic we study is the v-number of homogeneous ideals, which is a new invariant recently studied by many researchers, in connection to Algebraic Geometry, projective Reed-Muller-type codes and minimum distance functions. Let $I \subset S$ be a homogeneous ideal of a standard graded polynomial ring. For each associated prime $\mathfrak{p} \in \mathrm{Ass}(I)$ there exists a homogeneous element $f \in S$ such that $(I : f) = \mathfrak{p}$. The $\mathrm{v}_{\mathfrak{p}}$-*number of* I, $\mathrm{v}_{\mathfrak{p}}(I)$, is defined as the least degree of such an element. Whereas, the v-*number of* I is defined as $\mathrm{v}(I) = \min_{\mathfrak{p}\in\mathrm{Ass}(I)} \mathrm{v}_{\mathfrak{p}}(I)$. Inspired by results of Brodmann, Ratliff, Cutkosky, Herzog, Trung, Kodiyalam, and many others, about the asymptotic behaviour of powers of homogeneous ideals, we investigate the asymptotic behaviour of the function $\mathrm{v}(I^k)$. In one of our main results [7, Theorem 3.1] we prove that $\mathrm{v}(I^k) = \alpha(I)k + b$ is a linear function for all $k \gg 0$, where $\alpha(I)$ is the initial degree of I and b is a suitable integer. The v-number of monomial ideals, in connection with the famous Simon conjecture, was investigated in [5].

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Decompositions of powers of quadratic forms

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A Waring decomposition is an expression of a homogeneous polynomial as a sum of powers of linear forms. The minimum possible number of addends in such a decomposition is classically known as the Waring rank, or simply the rank, of the polynomial. We analyze the problem of determining the rank of the powers of quadratic forms, for which there are several examples of decompositions in the literature. We approach this problem from a contemporary perspective and provide some estimates for the rank value.

Abstract of the Ph.D. thesis

Secant varieties of Spinor varieties and of other generalized Grassmannians

Vincenzo Galgano

This thesis provides results on long-standing problems in the theory of Tensor Decomposition, namely on the problems of identifiability and singularity of points in the secant varieties of lines of both Grassmannians and Spinor varieties. Possible generalizations to other homogeneous spaces are also considered. The main spirit of this thesis is to investigate such topics from a representation-theoretical perspective, considering the action of groups on such varieties.

The key idea giving the kick-off to the whole thesis is that the action of a group G on a homogeneous variety X (SL_N for the Grassmannian $\mathrm{Gr}(k,N) \subset \mathbb{P}(\bigwedge^k \mathbb{C}^N)$, and Spin_{2N} for the Spinor variety $\mathbb{S}_N^+ \subset \mathbb{P}(\bigwedge^{ev} \mathbb{C}^N)$) leaves the secant variety of lines $\sigma_2(X)$ (and the tangential variety $\tau(X)$) invariant, which then splits into G–orbits, actually finitely many ones. Moreover, the properties of identifiability and singularity are invariant under such an action, and in particular the singular locus of the secant variety of lines is union of orbits. This implies that it is enough to check such properties for only one representative of each orbit.

First, we describe the poset (partially ordered set) of G–orbits in $\sigma_2(\mathrm{Gr}(k,N))$ and $\sigma_2(\mathbb{S}_N^+)$, exhibiting orbit representatives and computing the dimensions of the orbit closures.

Next, we completely solve the problems of identifiability and tangential-identifiability of points in the two secant varieties of lines. The results for Grassmannians are obtained by considering suitable wedge-multiplication maps between different fundamental SL_N–representations. The case of Spinor varieties is a little bit trickier and it is solved via a Spin_{2N}–equivariant map between Spin_{2N}–representations which we call *Clifford apolarity*.

We go one step further by also determining the 2*–nd Terracini locus* $\mathrm{Terr}_2(X)$, namely the locus of pairs of points of X whose tangent spaces have non-trivial intersection.

The previous results allow us to study the singular locus of $\sigma_2(\mathrm{Gr}(k,N))$ and $\sigma_2(\mathbb{S}_N^+)$. For Grassmannians, we prove that it coincides with closure of the unique non-identifiable orbit, correcting a previous statement on $\mathrm{Gr}(3,7)$ in [AOP12]. For Spinor varieties, we prove that the singular locus contains such closure and it is contained in the tangential variety: however, we conjecture that the first containement is an equality, as for the Grassmannian case.

The Grassmannian and Spinor variety cases share the same orbit posets and the same results on identifiability. Quite remarkably, we partially deduce a poset of G–orbits in the secant variety of lines of the other cominuscule varieties which is similar to the one obtained for Grassmannians and Spinor varieties. Finally, we exhibit an example of a non-cominuscule variety (i.e. the C_4–type isotropic Grassmannian $\mathrm{IG}(3,8)$) for which such a poset graph fails, showing that the orbits in the tangential variety are not totally ordered.

The results on Grassmannians have been obtained in the joint work [GS24] with Reynaldo Staffolani, while the ones on Spinor varieties are collected in the paper [Gal23].

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THERMAL INSTABILITY IN TWO-TEMPERATURE DARCY-BRINKMAN MODEL FOR ANISTROPIC POROUS LAYER: CATTANEO EFFECT IN THE SOLID

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In the field of hydrodynamics instability, the qualitative analysis of solutions in convection problems in porous media attract the attention of many researchers whose desire is to give a better understanding of the fluid motion, how and when the solution bifurcates and the origin of new stable solutions in time.

The classical model for convection in porous media is named after two scientists Darcy and Bénard who were the first to be interested in modeling fluid motion in porous media and thermal convection. As first attempt, significant approximations are introduced for a better tractability of the mathematical model. It is assumed the validity of a local thermal equilibrium (LTE) between fluid and solid matrix modeled by a single temperature equation employing the Fourier constitutive law for heat flux. However, the LTE condition may be inadequate in those physical frameworks where either fluid and solid thermal conductivities are very different or the portion of pores is much larger than the solid one.

Hence, in these frameworks it is necessary to assume that fluid and solid matrix are in local thermal nonequilibrium (LTNE). As a consequence, a two-temperature model is introduced to describe heat exchanges between the two phases. Moreover, it is well-known that the classical Fourier law, relating the heat flux to the gradient of temperature, leads to a paradox. According to the Fourier's law the evolution of temperature is governed by a parabolic heat equation. This equation implies an infinite speed of heat diffusion. To overcome the paradox, Cattaneo [1] in 1948 proposed a non-Fourier law by introducing a thermal relaxation time that leads to a hyperbolic heat equation and, consequently, to heat waves that propagates with finite speed. This correction is normally small and therefore difficult to detect experimentally. However, over the years, many scientists managed to prove the existence of thermal waves, both in rigid conductors at micro-scale and in fluids [2], [3].

This talk addresses a qualitative analysis of convective instability in a two-temperature Darcy-Brinkman model for an anisotropic porous layer saturated by a Newtonian fluid, where the Cattaneo's law is employed to model heat diffusion in the solid and the Fourier's law is retained for the fluid. It is proved that the presence of a thermal relaxation time (Cattaneo effect) leads to the occurrence of a new mode, which destabilises the basic solution, leading to oscillatory Hopf bifurcation.

The talk is based on a joint work [4] with F. Capone.

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Sharp second order regularity for widely degenerate elliptic equations

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Abstract

We present regularity results for local weak solutions to widely degenerate elliptic equations of the form

$$-\text{div}\left((|Du|-1)_+^{p-1} \frac{Du}{|Du|} \right) = f \quad \text{in } \Omega, \tag{1}$$

where $p > 1$ and Ω is an open subset of $\mathbb{R}^n$, with $n \geq 2$. Assuming either f belongs to the Besov space $B^{(p-2)/p}_{(p-1)/p,1,loc}(\Omega)$, if $p > 2$, or to the Lebesgue space $L^{\frac{np}{n(p-1)+2-p}}_{loc}(\Omega)$, if $1 < p \leq 2$, we show that $\mathcal{G}_{\alpha,p}\left((|Du|-1)_+\right) \in W^{1,2}_{loc}(\Omega)$, for every $\alpha \geq \frac{p+1}{2(p-1)}$, where we set

$$\mathcal{G}_{\alpha,p}(t) := \int_0^t \frac{s^{\frac{p-1+2\alpha}{2}}}{(s+1)^{\frac{1+2\alpha}{2}}}\, ds \qquad \text{for } t \geq 0.$$

The strategy is based on an integration by parts in fractional sense, together with a generalization to Besov spaces of the Nečas' negative norm Theorem, which states that taking a fractional derivative of negative order of f_{x_j} gives a fractional derivative of positive order. Here, the main novelty is that we prove a higher differentiability result for a nonlinear function of the gradient of weak solutions to (1) under sharp conditions on the right-hand side. This allows us to establish the higher integrability of Du under the same minimal requirements on the datum f.

The results are obtained in collaboration with P. Ambrosio (University of Naples Federico II) and A. Passarelli di Napoli (University of Naples Federico II).

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Non-linear MRD codes from cones over exterior sets

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(joint work with Nicola Durante and Giovanni Longobardi)

Abstract

In the finite projective space $\mathrm{PG}(n-1, q^n)$, let $\mathcal{X}$ be a C_F^σ-set of an $(n-k+1)$-dimensional subspace Λ with vertices A and B and $\Lambda^\star$ be a $(k-3)$-dimensional subspace skew with Λ. In [2], it is shown that $\mathcal{C}$ is a union of $\{A, B\}$ and $q-1$ pairwise disjoint scattered $\mathbb{F}_q$-linear sets of rank n, say $\mathcal{X}_a$ for any $a \in \mathbb{F}_q^*$. Moreover, the line AB can be partitioned in $\{A, B\}$ and $q-1$ scattered $\mathbb{F}_q$-linear sets of rank n, say J_a for any $a \in \mathbb{F}_q^*$. Denote by $\mathcal{K}(\Lambda^\star, \mathcal{E})$ the cone with vertex $\Lambda^\star$ and base the set

$$\mathcal{E} = \left(\mathcal{X} \setminus \bigcup_{a \in T} \mathcal{X}_a\right) \cup \bigcup_{a \in T} J_a,$$

with $1 \in T \subset \mathbb{F}_q^*$. Then $\mathcal{K}(\Lambda^\star, \mathcal{E})$ gives rise to a new family of non-linear $(n, n, q; d)$-MRD codes for any $n \geq 3$, $2 \leq d \leq n-1$ and $d = n-k+1$. By choosing the parameters or by puncturing appropriately a code in this class, the codes constructed in [1, 3] and in [2] are re-obtained. Finally, an element in this family, if not equivalent to a generalized Gabidulin, is not equivalent to any non-linear MRD codes constructed by Otal and Özbudak in [4].

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On the representability of actions of non-associative algebras

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It is well known that, in the semi-abelian category **Lie** of Lie algebras over a field $\mathbb{F}$, algebra actions are represented by derivations. This means that the category **Lie** is *action representable* and the representing object, which is called the *actor*, is the Lie algebra of derivations. The notion of action representable category has proven to be quite restrictive: for instance, if a non-abelian variety $\mathcal{V}$ of non-associative algebras over an infinite field $\mathbb{F}$, with $\mathrm{char}(\mathbb{F}) \neq 2$, is action representable, then $\mathcal{V} = \mathbf{Lie}$. More recently G. Janelidze introduced the notion of *weakly action representable category*, which includes a wider class of categories, such as the variety **Assoc** of associative algebras and the variety **Leib** of Leibniz algebras.

In this talk we show that for an *algebraically coherent* and *operadic* variety $\mathcal{V}$ and an object X of $\mathcal{V}$, it is always possible to construct a *partial algebra* $\mathcal{E}(X)$, called *external weak actor* of X, and a natural monomorphism of functors

$$\tau \colon \mathrm{Act}(-, X) \rightarrowtail \mathrm{Hom}_{\mathbf{PAlg}}(U(-), \mathcal{E}(X)),$$

where **PAlg** is the category of partial algebras over $\mathbb{F}$ and $U \colon \mathcal{V} \to \mathbf{PAlg}$ denotes the forgetful functor. The pair $(\mathcal{E}(X), \tau)$ is called *external weak representation* of the functor $\mathrm{Act}(-, X)$. Moreover, for any other object B of $\mathcal{V}$, we provide a complete description of the morphisms $(B \to \mathcal{E}(X)) \in \mathrm{Im}(\tau_B)$, i.e. of the homomorphisms of partial algebras which identify the actions of B on X in $\mathcal{V}$.

Eventually, we give an application of this construction in the context of varieties of *unital* algebras: we prove that, if $\mathcal{V} = \mathbf{Alt}$ is the variety of *alternative* algebras and X is a unital alternative algebra, then $\mathcal{E}(X) \cong X$ is the actor of X. In other words, unital alternative algebras, such as the algebra $\mathbb{O}$ of *octonions*, have representable actions.

This is joint work with Alan S. Cigoli (*Università degli Studi di Torino*, Italy), Xabier García Martínez (*Universidade de Vigo*, Spain), Giuseppe Metere (*Università degli Studi di Palermo*, Italy), Tim Van der Linden and Corentin Vienne (*Université catholique de Louvain*, Belgium).

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Title of the talk. A single player and a mass of agents: a pursuit evasion-like game

Authors. Fabio Bagagiolo, Rossana Capuani, Luciano Marzufero

Abstract. We study a finite-horizon differential game of pursuit-evasion like, between a single player and a mass of agents. The player and the mass directly control their own evolution, which for the mass is given by a first order PDE of transport equation type. Using also an adapted concept of non-anticipating strategies, we derive an infinite dimensional Isaacs equation, and by dynamic programming techniques we prove that the value function is the unique viscosity solution on a suitable invariant subset of a Hilbert space.

Rigidity results for the Robin $p-$Laplacian

Alba Lia Masiello

Abstract

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded, open and Lipschitz set and let f be a positive function. We consider the following problem

$$\begin{cases} -\Delta_p u := -\mathrm{div}(|\nabla u|^{p-2}\nabla u) = f & \text{in } \Omega \\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu} + \beta |u|^{p-2}u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where ν is the unit exterior normal to $\partial\Omega$ and $\beta > 0$, and its symmetrized version

$$\begin{cases} -\Delta_p v = f^\sharp & \text{in } \Omega^\sharp \\ |\nabla v|^{p-2}\dfrac{\partial v}{\partial \nu} + \beta |v|^{p-2}v = 0 & \text{on } \partial\Omega^\sharp, \end{cases} \tag{2}$$

where $\Omega^\sharp$ is the ball centered at the origin with the same measure of Ω.

In [1, 2] the authors prove a comparison á la Talenti between the solutions to equations (1) and (2). In particular, they prove

$$\|u\|_{L^{pk,p}(\Omega)} \leq \|v\|_{L^{pk,p}(\Omega^\sharp)}, \qquad \forall\, 0 < k \leq \frac{n(p-1)}{(n-2)p+n}, \tag{3}$$

and in the case $f \equiv 1$, they prove

$$\|u\|_{L^{pk,p}(\Omega)} \leq \|v\|_{L^{pk,p}(\Omega^\sharp)}, \qquad \forall\, 0 < k \leq \frac{n(p-1)}{(n-2)p+n}, \qquad \forall p > 1, \tag{4}$$

where $\|\cdot\|_{k,q}$ is the Lorentz norm of a measurable function.

In this seminar, following what we study in [3], we are interested in characterizing the equality cases in (3) and (4), proving that these estimates are rigid, i.e. the equality case can occur only in the symmetric setting.

This talk is based on joint works with Vincenzo Amato, Andrea Gentile and Gloria Paoli.

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Regularity for almost minimizer of a one-phase Bernoulli-type functional in Carnot Groups of step two

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In this talk, we focus on the regularity issues of almost-minimizers of Bernoulli-type energy functionals in Carnot groups of step 2. In the Euclidean setting, the regularity of minimizers of one-phase Bernoulli functional was deeply studied after the pioneering work of Alt and Caffarelli, [1]. More recently, also the regularity of almost-minimizers associated with Bernoulli-type functional has been investigated, in [3, 4]. In general, since almost-minimizer satisfies only a variational inequality, but not a partial differential equation, the main trouble in facing the regularity of such objects is the lack of a monotonicity formula, as minimizers do. In this perspective, in [4] is provided a different approach, based on non-variational techniques.

On the other hand, the one-phase free boundary problem arising from the minimization of the Bernoulli-type functionals may be stated even in the non-commutative setting of Carnot groups. Thereby, issues concerning the regularity of the minimizer and their free boundary immediately arise. Carnot groups, which represent the tangent model of a general sub-Riemannian manifold and are the natural spaces to model some physical systems with non-holonomic constraints have been deeply investigated from different perspectives. However, the approach to the regularity theory of free boundary problems based on the monotonicity formulas, well-understood in the Euclidean setting, seems to be hard to generalize to the non-commutative setting of Carnot groups. As proved recently in [5] in the Heisenberg group, the natural counterpart tool given by an intrinsic Alt-Caffarelli-Friedman monotonicity formula seems to fail. Furthermore, the approach based on viscosity tools appears more challenging due to the theoretical existence of characteristic points on the free boundary. Nevertheless, with our approach, we overcome these difficulties.

In particular, the main contribution we want to present is the Lipschitz regularity of almost-minimizer of a one-phase Bernoulli-type functional in Carnot Groups of step two. Such a result extends the regularity result proved in [8] to the more general setting of almost-minimizer in a general Carnot group of step 2. In addition, the tools and techniques employed in this paper are different and appear more flexible. We managed to obtain our main result by applying the inner regularity results studied in Carnot groups of step two in [2].

Some of the results presented are obtained in collaboration with F. Ferrari (University of Bologna) and N. Forcillo (Michigan State University) and will be part of my PhD thesis.

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Mathematical induction and chains of inferences: A cognitive-didactical analysis

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Abstract

The relevance of mathematical induction as a proving scheme in mathematics is beyond doubt. Its role, fundamental and foundational, is central both from an epistemological and practical standpoint in the mathematician's activity. However, research in mathematics education unanimously highlights how it is an extremely delicate, if not problematic, topic from a didactical perspective, with difficulties among students observed across various school levels.

The study presented in this talk, which is part of a broader research on mathematical induction from a cognitive and didactical standpoint developed during my doctoral studies, aims to investigate the understanding of this proving scheme by university students. Specifically, by adopting two specific theoretical frameworks in mathematics education, the *APOS theory* [1] and the framework of *intuitions* according to Fischbein [2], some central cognitive processes related to this proving scheme will be described and analysed. These include the construction of chains of logical inferences from the base case to certain natural numbers and the transition from constructing these chains to imagining them without explicitly constructing them. A paradigmatic example of one of these chains is observable in the classic informal justification of why mathematical induction 'works': from the propositions $P(n_0)$ and $P(n_0)\rightarrow P(n_1)$, it follows $P(n_1)$ by *modus ponens*; from $P(n_1)$ and $P(n_1)\rightarrow P(n_2)$, it follows $P(n_2)$ by *modus ponens*, and so on, until any $n>n_0$.

To investigate the extent to which university students have interiorized the construction of such chains of inferences, I conducted an empirical study based on a questionnaire involving 306 students from various years and degree courses. In the talk, I will present an analysis of the results and discuss some conclusions that emerge from these findings, including: (1) The process of constructing chains of logical inferences tends to be generally more problematic when the distance from the base of induction increases or when it involves indirect inferences (such as *modus tollens*); (2) Students may find it challenging to accept the validity of mathematical induction, understood as a proving scheme involving a generic predicate, without knowing the specific property to which the generic predicate refers.

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Approximation by nonlinear multivariate sampling Kantorovich operators in some functional spaces

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In the present talk, we will focus our attention on a special class of nonlinear operators, namely the so-called *nonlinear multivariate sampling Kantorovich operators* in the general setting of modular spaces. The analysis of such operators goes beyond their mathematical significance and finds practical applications in various fields. One important example can be furnished in Signal Processing, when one has to describe a signal generated by an earthquake, an explosion, or also some nonlinear transformations generated by signals that, during their filtering process, produce new frequencies. A wide letterature can be found in [7, 6, 1, 8, 5].

In [2, 4] we establish the rate of convergence through both quantitative and qualitative analysis in the space of bounded and uniformly continuous functions and in the setting of Orlicz spaces. In this respect, a crucial role is played by the basical properties of the modulus of continuity and the modulus of smoothness, respectively.

Further, in [3] we provide convergence results in the broader setting of modular spaces, under suitable assumptions and together with a modular inequality. In [4] we prove some quantitative estimates within modular spaces by using a direct approach. This extends the field of applications and enables us to give a unifying approach to several settings of approximation problems. In fact, modular spaces include Musielak-Orlicz spaces, which contain, for instance, weighted-Orlicz spaces and Orlicz spaces, as well as spaces of functions equipped by modulars that are not of integral type.

Finally, by using some special kernels, several examples of nonlinear multivariate sampling Kantorovich operators are provided.

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A COMBINATORIAL INTERPRETATION OF THE PRIMARY DECOMPOSITION OF BINOMIAL IDEALS ATTACHED TO POLYOCOLLECTIONS

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Combinatorial Commutative Algebra has been providing an intensive area of research since the pioneering work of Richard Stanley ([3]). As the name suggests, it comes from the intersection of two research areas, which are Commutative Algebra and Combinatorics. The purpose is to investigate the main algebraic properties of rings and ideals which can be defined from combinatorial objects, like posets, graphs, simplicial complexes and many others.

In 2012 a new topic in Combinatorial Commutative Algebra has emerged by a work of Ayesha Asloob Qureshi. In [2] she establishes a connection between collections of cells and Commutative Algebra, assigning to every collection $\mathcal{P}$ of cells the ideal of the inner 2-minors of $\mathcal{P}$ in a suitable polynomial ring $S_{\mathcal{P}}$ (see [2]). This ideal $I_{\mathcal{P}}$ is called the *inner 2-minor* ideal of $\mathcal{P}$ and $K[\mathcal{P}] = S_{\mathcal{P}}/I_{\mathcal{P}}$ is said the *coordinate ring* of $\mathcal{P}$. In particular, if $\mathcal{P}$ is a polyomino, that is a collection of cells where the squares are joined edge by edge, then $I_{\mathcal{P}}$ is called the *polyomino ideal* of $\mathcal{P}$.

The aim of the research is to study the main algebraic properties of $K[\mathcal{P}]$ depending on the shape of $\mathcal{P}$. This has been giving many exciting challenges and one of the most interesting is to provide a combinatorial description of the primary decomposition of $I_{\mathcal{P}}$.

In [1] we show that for studying the primary decomposition of the polyomino ideals, we should consider a larger class of binomial ideals. This class is related to a new combinatorial object, called *polyocollection*, which generalizes the concept of collection of cells and polyomino. We introduce a binomial ideal attached to a polyocollection, generalizing the ideal associated to a collection of cells in [1]. We provide a characterization of the primality of such a binomial ideal in terms of the lattice ideal attached to the polyocollection and we give a primary decomposition of the radical of that ideal using the so-called admissible sets and the lattice ideals of some suitable polyocollections. Finally, we give a detailed description of the minimal primary decomposition of a particular class of polyominoes, namely closed path polyominoes. We show that the polyomino ideal is the intersection of only two minimal prime ideals and both minimal prime ideals have a very nice combinatorial interpretation in terms of the so-called zig-zag walks and of the vertices in a so-called necklace.

Joint work with Carmelo Cisto and Dharm Veer.

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Enhanced discretizations for PDEs models from applications

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ABSTRACT

The use of Partial Differential Equations (PDEs) represents one of the most common strategies for modeling several phenomena. We are particularly interested in the numerical solution of PDEs models coming from real applications, such as corrosion of materials (Frasca-Caccia et al., Math. Model. Nat. Phenom., 2023), sustainability (Maldon et al., Entropy, 2020), vegetation (Eigentler et al., Bull. Math. Biol., 2019). The numerical treatment of these models is not trivial, since a related spatial discretization, performed e.g. via classical finite differences, finite elements, spectral methods, often leads to Initial Value Problems (IVPs) characterized by large size and high stiffness. To deal with this last issue, the use of very dense temporal grids is required. Thus, there is a need to resort to enhanced methods, capable of providing a numerical solution accurately and efficiently, perhaps also exploiting a-priori known properties of the model, such as any positivity or oscillating trend, asymptotic stability, and so on.

This talk focuses on enhanced linearly implicit methods obtained by stabilizing explicit numerical schemes through TASE (Time-Accurate and highly-Stable Explicit) operators [1], for PDEs models from applications. In particular, starting from classical TASE-Runge-Kutta (RK) methods [2], we first propose a generalization that allows solving the underlying linear systems without updating the inherent coefficient matrices at each step, studying in detail the related consistency and stability [4]. Subsequently, to further improve the classical TASE-RK methods, starting from them we derive two new classes of numerical schemes: TASE-peer methods [3], which can be parallelized; TASE-W methods [6], which have better accuracy and stability properties than TASE-RK, and require the solution of a lower number of linear systems at each integration step. Finally, exploiting connections between TASE operators and exponential integrators, we derive new generalized non-standard discretizations for a reaction-diffusion PDEs model for vegetation growth in the African Savannah [5], capable of preserving the positivity of the solution and the oscillating trend even choosing non-dense spatial and temporal grids. Numerical results testify the efficiency of the proposed methods.

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Positive and High Order Numerical Methods for Integro-differential Epidemic Models

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The age-of-infection integro-differential model, originally proposed by Kermack and McKendrick [4, 5], offers a versatile yet computationally challenging framework for a comprehensive description of various infectious disease outbreak scenarios. The PhD thesis [7], encompassing some joint works with Eleonora Messina (University of Naples Federico II) and Antonia Vecchio (Institute for Computational Application Mauro Picone), addresses the challenges posed by the long-time simulation of the aforementioned epidemic model and of its extensions [1, 2] through the introduction of dynamically consistent numerical methods. Specifically, unconditionally positive and high-order schemes based on Non-Standard Finite Difference [6] and Direct Quadrature [3] methods are here devised. Moreover, a thorough investigation of the asymptotic behavior of the solution to non-linear implicit Volterra discrete equations is carried out. Applying these outcomes to the numerical methods for age-of-infection models, we rigorously demonstrate the convergence of the discrete final size of the epidemic and establish our schemes as reliable tools for assessing the long-term behavior of the outbreak.

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Approximation and regularization properties of Durrmeyer sampling type operators in functional spaces

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The theory of sampling series, both in one and several variables, holds a significant place in Approximation Theory, in view of its many and various applications, particularly in Signal and Image Processing. The focus of this talk is to deliver an overview on the approximation properties, including regularization results, for a semi-discrete version of sampling operators, represented by the so-called *Durrmeyer sampling type operators* (DSO) [9, 2]. Originating from a sharp modification of Bernstein polynomials, these operators play a crucial role in extending the celebrated Weierstrass approximation theorem to broader functional spaces. Notably, they also expand other well-known families of sampling operators, including both generalized [4] and Kantorovich [1] types.
We study the regularization properties of DSO by using a distributional approach [8], we provide convergence results in several functional spaces [5], and we estimate the order of approximation via a quantitative and qualitative analysis [6]. To achieve this, we employ a unifying method that yields approximation results applicable to a broad spectrum of functions, even those not necessarily continuous, so as to obtain advantages from both an applications and theoretical point of view. The main framework is indeed represented by the general setting of Orlicz spaces [3], introduced as a natural extension of the well-known Lebesgue spaces. Furthermore, we expand the central approximation properties described here in the multidimensional setting [7], thereby establishing the groundwork for potential future applications in Image Processing.
Finally, the main results presented herein will be also discussed via several instances of kernels, such as Fejér, B-spline and Bochner-Riesz kernel.

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The second problem of applicability in the work of Julius Weingarten

Rachele Rivis

The aim of my thesis is to focus on some important moments in the development of the classic infinitesimal theory of surfaces in the eighteenth and nineteenth centuries that concern the second problem of applicability, i.e., the determination of all surfaces locally isometric to a given one.

In the talk, we will describe the important contributions on this subject offered by Julius Weingarten.

Weingarten's interest in applicability arose early in his career when, in (Weingarten 1861) and (Weingarten 1863), he launched the $W-$surface theory. Then, in (Weingarten 1884), he identified some critical points that affected previous attempts to the problem and gradually started to develop a new approach. At first, his focus was mainly on the determination of new complete classes of applicable surfaces. Only later, probably on Darboux's advice, he shift his attention to a general method for the second problem of applicability. Weingarten succeeded in deducing a second-order PDE of Monge-Ampère type, whose solutions give all the surfaces applicable to a given one. In contrast to previous results, it is integrable by known methods in all the cases in which complete classes of applicable surfaces had already been found. These achievements were awarded the 1894 Grand Prix des Mathématiques of the Académie des Sciences in Paris and published in (Weingarten 1897).

This process is documented in numerous letters that Weingarten sent to his colleague Luigi Bianchi, which were published in (Bianchi 1959). An in-depth reading of this correspondence along with an analysis of the content of Weingarten's published results on applicability allows us to verify Bianchi's words: *"by studying his particular procedures more carefully, Weingarten succeeded in obtaining a radical transformation of the applicability equation"* (Bianchi 1910, p. 224).

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PhD thesis abstract - Nicholas Rungi
Pseudo-Kähler geometry of Hitchin representations and convex projective structures

In this thesis we study the symplectic and pseudo-Riemannian geometry of the $\mathbb{P}\mathrm{SL}(3,\mathbb{R})$-Hitchin component associated with a closed orientable surface, using an approach coming from the theory of symplectic reduction in an infinite-dimensional context.
In the case where the closed surface is homeomorphic to a torus, for each choice of a smooth real function with certain properties, we prove the existence of a pseudo-Kähler metric on the deformation space of properly convex projective structures. Moreover, we define a circle action and a $\mathrm{SL}(2,\mathbb{R})$-action on the aforementioned space, which turn out to be Hamiltonian with respect to our symplectic form, and we give an explicit description of the moment maps. Then, we study the symplectic geometry of the deformation space as a completely integrable Hamiltonian system, and we find a geometric global Darboux frame for the symplectic form using the theory of complete Lagrangian fibrations.
In the case of genus $g \geqslant 2$ we define a mapping class group invariant pseudo-Kähler metric on the Hitchin component, by using a general construction of Donaldson. The complex structure is exactly the one coming from the identification with the holomorphic bundle of cubic differentials over Teichmüller space. In particular, we prove that Wang's equation for hyperbolic affine spheres in $\mathbb{R}^3$ has an interpretation as moment map for the action of an infinite-dimensional Lie group.

The thesis consists of three research papers listed below:

Rungi, N., Tamburelli, A. "The $\mathbb{P}\mathrm{SL}(3,\mathbb{R})$-Hitchin component as an infinite-dimensional pseudo-Kähler reduction", arXiv:2306.02699.

Rungi, N., Tamburelli, A. "Global Darboux coordinates for complete Lagrangian fibrations and an application to the deformation space of projective structures in genus one", arXiv:2208.05336 (to appear in *Journal of Symplectic Geometry* 2024).

Rungi, N., Tamburelli, A. "Pseudo-Kähler geometry of properly convex projective structures on the torus", arXiv:2112.08979 (to appear in *Journal of Geometric Analysis* 2024).

A geometric perspective on condition numbers for tensor decomposition

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Given a nondegenerate projective variety $X \subset \mathbb{P}^N$, a set of r points of X belongs to the r-**Terracini locus** of X if the span of the r tangent spaces of X at these points are linearly dependent, i.e. if

$$\dim\langle T_{p_1}X, \ldots, T_{p_r}X\rangle < r(\dim X + 1) - 1,$$

where $p_1, \ldots, p_r$ are regular points of X. Terracini loci have been introduced in collaboration with Edoardo Ballico and Alessandra Bernardi [1] in the context of tensor decomposition and simultaneously in [3] in the more general context of projective varieties. The study of these objects has been one of the main focuses of my Ph.D. [10] and since their introduction, Terracini loci have started to be extensively investigated (see, e.g., [2, 6, 8, 9]) and are becoming an established topic in the theory of tensor decomposition and projective geometry.

Born from the desire of understanding the behaviour of non generic points, Terracini loci in the context of tensor decomposition are more than just an interesting purely geometric object because they share a deep connection with condition numbers. The **condition number** of a function at a point measures the rate of error that happens to an output element conditioned to a small change on the element in the domain. A problem is *well conditioned* if it has a small condition number. Working with an ill conditioned problem (that has an *high* condition number) implies that the problem itself is sensitive to small perturbations, which is a critical property to deal with when attempting to implement a function into a machine. Hence, Terracini loci of Segre varieties (or of other tensor related varieties) are involved when measuring the sensitivity of a tensor rank decomposition (see, e.g., [4, 5, 7]). In particular, Terracini loci essentially characterize all the tuple of rank one tensors that have an infinite condition number, and hence for which it is impossible to uniquely recover the elementary tensors parametrizing the decomposition. For such data, unique recovery is numerically infeasible, so the knowledge of Terracini locus in the tensor decomposition context is crucial for any numerical implementation.

In this talk, I will introduce Terracini loci in the context of tensor decomposition, carefully explaining their connection to condition numbers. Time permitting, I will present some of the latest results involving these extremely interesting objects.

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Juan M Santiago Suárez
(joint work with Matteo Viale)

Model theory for set theory.

Let X be a structure and x an element of it. One way to understand the properties of x is to work out the properties of the substructure generated by it. Our research is motivated by this question in the area of set theory.

We denote V the universe of all sets. In 1938 [1] Kurt Gödel introduced the smallest class sized model of set theory, it is denoted L and contains all constructible sets. These are the sets that one can obtain through an iteration of simple operations like complements or cartesiens products. Later on, András Hajnal generalized this construction relative to any set A. This construction produces the model $\mathrm{L}(A)$ of all constructible sets relative to the oracle A. Hence, we have a precise notion of what is the substructure generated by a set A in the context of set theory.

A central question is understanding how these models relate and differ from each other. For example, even if choice is supposed to be true in the whole universe V, it might not hold in models as important and canonical as $\mathrm{L}(\mathbb{R})$. Nonetheless, choice does hold in the smaller model L and from large cardinal axioms it holds in the bigger model $\mathrm{L}(P(\omega_1))$:

$$\mathrm{L} \subset \mathrm{L}(\mathbb{R}) = \mathrm{L}(P(\omega)) \subset \mathrm{L}(P(\omega_1)).$$

Once we have a family of structures $(\mathrm{L}(A) : A \in V)$, we need tools to understand how they relate. In set theory, the main tool is called forcing and was introduced by Paul Cohen [2]. Forcing allows to extend models of set theory using approximations existing in the ground model. Hence, we can made formal natural questions such as: is V a forcing extension of some inner model? is $\mathrm{L}(A)$ a forcing extension of $\mathrm{L}(B)$ for B smaller than A? This last result was studied by Hugh Woodin in [3] where the $\mathbb{P}_{\max}$ technique and the axiom $(*)$ are introduced. Under large cardinals, $\mathrm{L}(P(\omega_1))$ is a forcing extension of $\mathrm{L}(P(\omega))$ in which the axiom of choice is recovered.

Our goal is to understand all these results from a model theoretic perspective. We call the theory of a model the set of all formulas or properties that are true in it. For example, the sentence stating the existence of solutions for all polynomial equations is in the theory of $\mathbb{C}$, but not in that of $\mathbb{R}$. Model theory studies structures by understanding the properties of their theories. Our work has related both the set theoretic perspective with the model theoretic one, culminating with the following result:

The theory of $\mathrm{L}(A)$ is "tame" whenever A is a "good" family of sets of reals,

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Gino Fano (1871-1952): patrimonio, ricerca e insegnamento

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Mentre le traiettorie scientifico-professionali dei principali geometri della Scuola italiana di geometria algebrica, i loro risultati e la cronologia della Scuola sono ormai consolidati in letteratura, minori sono i contributi volti a fornire una visione più precisa e approfondita sugli altri membri e a studiare le fonti materiali dell'impresa scientifica di questo gruppo sociale, di primaria importanza per la storia della matematica in Italia. Il case-study di Gino Fano, oggetto della mia tesi di dottorato, rappresenta un contributo in entrambe le direzioni. Da una parte, infatti, nonostante i principali tratti della biografia di Fano fossero noti, diversi erano gli aspetti della sua attività scientifica, didattica e istituzionale ancora da esplorare. Dall'altra parte, non era stato analizzato nel dettaglio il corposo fondo archivistico di Fano, custodito presso la Biblioteca Speciale di Matematica dell'Università di Torino. Partendo dal presupposto che l'attività matematica è fortemente situata, il concetto di patrimonio (nella sua duplice accezione materiale e immateriale) costituisce un'efficace categoria interpretativa per lo storico della matematica.

Un interessante esempio in tal senso è costituito da uno dei contributi matematici di Fano analizzati nella tesi: le cosiddette trasformazioni birazionali di contatto del piano. In termini moderni, denotando con $\mathbb{P}\Omega^1_{\mathbb{P}^2}$ il proiettivizzato del fibrato cotangente di $\mathbb{P}^2$, si tratta di automorfismi birazionali a di $\mathbb{P}\Omega^1_{\mathbb{P}^2}$ tali che $p \cdot a = b \cdot p$, dove $p\colon \mathbb{P}\Omega^1_{\mathbb{P}^2} \to \mathbb{P}^2$ è la proiezione naturale e $b \in Bir(\mathbb{P}^2)$.

A partire dallo studio delle minute manoscritte della conferenza che Fano, esule a Losanna in seguito ai provvedimenti razziali, tenne al *Cercle Mathématique* nel febbraio del 1944 si metterà in luce l'evoluzione del suo percorso di ricerca su questo tema tra il 1925 e il 1947. Emergerà come gli studi sulle trasformazioni birazionali di contatto del piano non siano soltanto fortemente correlati alle varietà tridimensionali, ancora oggi legate al nome di Fano (*Fano threefolds*) ma rappresentino anche una specifica operazione culturale: Fano infatti utilizza i metodi e gli strumenti di indagine tipici della Scuola italiana per dare una nuova veste a un oggetto matematico sorto all'interno di un retroterra culturale differente, quello della geometria differenziale di S. Lie, e preso in esame soltanto da L. Autonne dal punto di vista analitico. Per far ciò, Fano attinge a piene mani dal patrimonio geometrico italiano: fa ricorso a elementi caratteristici della Scuola di C. Segre e fornisce una nuova interpretazione "geometrica" degli oggetti in questione. Saranno inoltre prese in considerazione le condizioni materiali in cui Fano svolge la propria attività di ricerca in Svizzera, così come i progressivi cambiamenti nella letteratura di riferimento. Si porrà infine l'accento sull'impegno di Fano per la promozione all'estero di quell'"edificio grandioso" e di quello "stile caratteristico" della Scuola italiana, una cui preziosa testimonianza è costituita proprio dalla conferenza di Losanna sulle trasformazioni birazionali di contatto del piano, al cui interno ben si coniugano divulgazione e ricerca.

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STABILITY OF COHERENT STRUCTURES IN DISSIPATIVE VARIANTS OF NONLINEAR SCHRÖDINGER-TYPE EQUATIONS

BORIS.SHAKAROV

The Hamiltonian Nonlinear Schrödinger (NLS) equation is an effective model widely used in several areas of physics. There is a considerable amount of cases where the classical NLS is augmented with a dissipative term, modeling mass or energy losses. This thesis analyses various dissipative NLS-type equations, with a focus on the well-posedness theory and the stability of some specific solutions, such as dissipative solitons, self-similar blow-up profiles, or stationary solutions. This work is divided into three parts.
In the first part, we consider the focusing, mass sub-critical nonlinear Schrödinger equation with the addition of a linear damping on a periodic domain. The existence and the orbital stability of several types of solitary waves are known in the Hamiltonian case. Under the damped dynamics, solitary waves can be stable only in a finite time interval proportional to the inverse of the damping coefficient. However, by modulating efficiently the solitary wave size according to the dissipative dynamics, we have retrieved a generalized notion of orbital stability for a family of solitons for all times. We prove this result for cnoidal and dnoidal wave profiles in the periodic setting.
In the second part, we will investigate the effect of nonlinear damping on the formation of singularities in finite time. It is well known that the focusing undamped NLS dynamics allows a finite time blow-up. The addition of the damping may prevent the blow-up, but the limiting case where the nonlinear damping has the same power as the power type nonlinearity was open in the literature. We provide a negative answer to this question, by showing the formation of self-similar blowing-up solutions for the damped dynamics. In particular, our result rigorously proves that, for mass-supercritical focusing and dissipative nonlinearity of the same power, the damping does not regularize the dynamics.
In the last part, we analyze two dissipative models that find their motivations in numerical studies of generalized NLS equations. Both models are examples of dynamical systems constrained on the manifold with fixed L^2 norm. This is achieved by adding a suitable nonlocal term to the equation, that balances the mass exchange with the external environment. The two models consist of nonlinear parabolic dynamics and a complex Ginzburg-Landau-type equation. We will study the local and the global well-posedness of the model, and we find also sufficient conditions under which the dynamics strongly converge for large times towards a stationary solution. We will characterize the $\omega-$limit set and find sufficient conditions on the data for a convergence to the ground state.

Date: January 2024.

All the works are done in collaboration with my Ph.D. advisor P. Antonelli (GSSI, L'Aquila). The parabolic model in the third part is also based on a work with Piermarco Cannarsa (Tor Vergata, Roma).

Empirical Density Estimation based on Spline Quasi-Interpolation with applications to Copulas clustering modeling

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Abstract

Density estimation is a fundamental technique employed in various fields to model and to understand the underlying distribution of data. The primary objective of density estimation is to estimate the probability density function of a random variable. This process is particularly valuable when dealing with univariate or multivariate data and is essential for tasks such as clustering, anomaly detection, and generative modeling. In this paper we propose the mono-variate approximation of the density using spline quasi interpolation and we applied it in the context of clustering modeling. The clustering technique used is based on the construction of suitable multivariate distributions which rely on the estimation of the monovariate empirical densities (marginals). Such an approximation is achieved by using the proposed spline quasi-interpolation, while the joint distributions to model the sought clustering partition is constructed with the use of copulas functions. In particular, since copulas can capture the dependence between the features of the data independently from the marginal distributions, a finite mixture copula model is proposed. The presented algorithm is validated on artificial and real datasets.

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Title and abstract of the thesis of the Ph.D. student Jacopo Tenan, from "Università degli Studi di Roma Tor Vergata"

Title of my Ph.D. Thesis: Volume preserving mean curvature flow in asymptotically flat 3-manifolds.

Abstract: The main purpose of my work is to generalize the well known approach by Huisken and Yau [4], who studied the *volume preserving mean curvature flow* (VPMCF) to construct a constant mean curvature (CMC)-foliation of asymptotically-Schwarzschild manifolds. The existence result for the foliation was later extended under more general assumptions by other authors, e.g. [5], [3], [6], by different methods which did not employ flows.
We prove long time existence and convergence to a constant mean curvature surface for the VPMCF of a round hypersurface in a $C^2_{\frac{1}{2}+\delta}$-asymptotically flat manifold with positive ADM-mass. The core of our proof is a suitable definition of the class of round surfaces, which involves integral norms of the traceless second fundamental form and of the oscillation of the mean curvature. We are then able to prove invariance of this class by a careful analysis of the time evolution of our integral quantities, combined with powerful recent results from the literature, such as [2] and the spectral analysis of the stability operator on almost round surfaces of [6].
As a consequence, we recover by a flow approach the existence of the CMC foliation under the same sharp hypotheses as in [6]. In particular, we do not need hypotheses on the decay of the derivatives of the Riemann curvature tensor of the ambient manifold, nor symmetry assumptions.

The integral approach also allows to study variation of the VPMCF, such as the volume preserving spacetime mean curvature flow, where the mean curvature in the velocity of the flow is replaced by the so called *spacetime mean curvature*, recovering the recent result of Cederbaum and Sakovich [1].

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II meeting UMI for Doctoral Students
Naples, June 13-14, 2024

Sampling Type Operators and Applications

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The talk is focused on some recent approximation results for the nonlinear version of Durrmeyer-sampling type operators and applications in the biomedical field. For what concerns the space of continuous functions, a pointwise and uniform convergence theorem for the nonlinear Durrmeyer-sampling type operator is provided. Moreover, approximation results in the general setting of Orlicz spaces are also discussed [3]. The above mentioned family of operators, introduced in [1] in their linear version and studied, e.g., in [5], represents an extension of other well-known families of sampling operators, including both generalized and Kantorovich ones [2,4]. Convergence and approximation results in multidimensional settings for the sampling Kantorovich operators, in their linear version, represent the theory at the basis of applications in the field of digital image reconstruction and processing [7]. The features of these operators, when implemented, enable an enhancement of the images to be reconstructed. Indeed, they act simoultaneously as rescaling algorithms, increasing the information content of the image, and as low-pass filters, attenuating potential disturbances caused by noise. For what concerns applications in the biomedical field, three different studies will be considered. A first one, in which the sampling Kantorovich operator, is used to process magnetic resonance images for the identification of biomarkers for Alzheimer's disease [6]. A second study developed on 13677 images from 15 patients affected by moderate/severe atheromatous disease of the abdominal aortic tract, consist in exploiting a procedure to extract the pervious lumen of the aorta artery from basal CT images (i.e., without contrast medium). Numerical indices of errors were computed and analyzed, together with a clinical evaluation, in order to test the validity of the proposed method [8]. Finally a study concerning the analysis of eye fundus images from both healthy and diabetic individuals, which consists in segmenting the superficial and profundus parts of the eye, in order to assess a cluster counting analysis, is also discussed [9].

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Modeling reacting gas mixtures: from kinetic theory to reaction-diffusion systems

Marzia Bisi, Giorgio Martalò, Romina Travaglini

Abstract

We propose the study of binary gas mixtures, specifically focusing on a polyatomic (diatomic) one and a monatomic, diffusing in a gaseous background, such as the atmosphere. The investigation concerns the effects of interactions among molecules, including elastic and inelastic scattering, coupled with reversible and irreversible chemical reactions. By adopting a time scaling approach, using the Knudsen number (ϵ), we establish a hierarchy of processes based on their time scales. More in detail, elastic scattering dominates, while other interactions occur at varying orders of ϵ. The rescaled kinetic Boltzmann equations are formulated, and through a subsequent hydrodynamic limit, we derive reaction-diffusion equations for the number densities of the gas components. Notably, in one of the time scaling proposed, a cross-diffusion term for the polyatomic species emerges, characterized by a non-constant function of the total density of the monatomic species and that of the monatomic one.

The discussion comprehends the exploration of Turing instability phenomena within the macroscopic systems obtained. We analyze the stability properties, setting the discussion upon microscopic features of the mixture, such as particle masses, collision frequencies of the Boltzmann operators, and particle internal energies. Condition of parameters are then found in such a way formation of spatial patterns for macroscopic densities can be observed by numerical simulations.

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Integral-geometric (in)equalities and Soap Bubble theorems for Sobolev-type domains

Paolo Valentini

University of L'Aquila

Meeting UMI for Doctoral Students, Naples, June 13-14, 2024

Abstract

We extend classical results of Differential Geometry such as Minkowski-Hsiung identities, Heintze-Karcher inequality and Alexandrov Soap Bubble theorems [1, 3, 4], to certain classes of domains in $\mathbb{R}^{n+1}$ whose topological boundary is representable locally as graphs of $W^{2,n}$-Sobolev functions. Our results are phrased in terms of the natural weak notion of curvature, associated with Sobolev graphs. A crucial point in the analysis is to construct a Legendrian cycle associated with these domains, exploiting Federer's theory of current, Fu's theory of Monge-Ampere functions [2] and fine properties of $W^{2,n}$-Sobolev functions.

This is a joint work with Mario Santilli (University of L'Aquila).

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SYZ mirror symmetry of solvmanifolds

Alessandro Vannini "Università dell'Aquila"

We will present an effective construction of non-Kähler supersymmetric mirror pairs in the sense of Lau, Tseng, and Yau starting from left-invariant affine structures on Lie groups. We will first discuss how mirror symmetry can be generalized to the non-Kähler setting using SU(n)-structures. Next, we will see how to obtain the required structures by intertwining the connection between affine and symplectic geometry with the theory of solvmanifolds. Applying this construction we will then find SYZ mirror symmetric partners of all known compact 6-dimensional completely solvable solvmanifolds that admit a semi-flat type IIA structure. Cohomology computations and relative non-Kähler Hodge diamonds will also be given. This is a joint work with L. Bedulli.

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Titolo della tesi: **La rinascita della Logica in Italia nella seconda metà del '900: aspetti storici e didattici.**
Autore: **Antonio Veredice**
Advisors: **Prof. Claudio Bernardi**, **Prof. Lorenzo Tortora de Falco**.
Discussa in data: **17 luglio 2023**

Abstract

All'origine dei vari indirizzi della ricerca in Logica Matematica che viene svolta attualmente in Italia c'è un preciso momento di nascita: la fondazione del Gruppo di Logica del CNR da parte di Ludovico Geymonat nel 1962. A questo momento storico è stato dato il nome di rinascita della Logica italiana per sottolineare la soluzione di continuità con la Logica di Peano di inizio secolo.

In questa tesi abbiamo cercato di ricostruire alcuni aspetti del percorso che ha portato la Logica Matematica in Italia, da una disciplina appannaggio di un ristretto gruppo di pionieri ad una vasta area di studio che interessa oggi un numero non trascurabile di ricercatori che contribuiscono in maniera importante alla ricerca internazionale. Per far ciò ci siamo avvalsi, oltre che degli strumenti tradizionali della ricerca matematica, anche di interviste a ricercatori che hanno vissuto direttamente la rinascita della Logica in Italia.

La seconda metà del '900 in Italia ha visto la Logica come crocevia tra Matematica, Filosofia e Informatica in un intreccio di rapporti e contaminazioni che hanno dato origine, in qualche caso, anche a controversie. Inoltre, attraverso la Logica, il mondo della ricerca scientifica e il mondo della scuola sono entrati in contatto e ciò ha dato vita a interessanti collaborazioni.

In questo lavoro abbiamo scelto alcuni temi di ricerca (in Algebra della Logica, Teoria della Dimostrazione e Teoria dei Modelli) e studiato i contributi di alcuni studiosi italiani in tali ambiti, cercando di far emergere quali sono stati gli snodi concettuali caratteristici della storia della Logica Italiana.

Ci siamo infine occupati di questioni che hanno legato l'evoluzione storica della Logica italiana ad ambiti che esulano dalla ricerca scientifica, come ad esempio l'introduzione di argomenti di Logica nei curricula scolastici di Matematica.

RATIONALITY OF FINITE GROUPS: GROUPS WITH QUADRATIC FIELDS OF VALUES

MARCO VERGANI

Abstract. The main topic of this seminar is families of groups that have a characterization of their integral central units inside the rational group algebra. Using representation theory it is possible to consider groups as acting over vector spaces in a natural way, relating the irreducibile actions to the field generated by the trace of the representation. Those fields give us a lot of information about the group itself. In this seminar we will focus on groups with field of values that are quadratic extensions of the rationals and we will define tools that allow us to detect how far the group is from a "rational" action.

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On the Security of MQ in the Multi-Instance Setting

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April 22, 2024

Abstract

In [1, 2], L. Bidoux and P. Gaborit introduced a new general technique to improve Zero-Knowledge (ZK) Proof-of Knowledge (PoK) schemes for a large set of well-known post-quantum hard computational problems such as the Syndrome Decoding, Permuted Kernel, Rank Syndrome Decoding, and the Multivariate Quadratic (MQ) problems. In particular, the idea of the authors is to use the structure of these problems in the multi-instance setting to minimize the communication complexity of the resulting ZKPoK schemes. The security of the new schemes is then related to new hard problems. In this talk, I will focus on the new multivariate-based ZKPoK and the corresponding new underlying problem: the so-called $\texttt{DiffMQ}_{\mathrm{H}}$. I will present a new efficient probabilistic algorithm for solving the $\texttt{DiffMQ}_{\mathrm{H}}$.

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GRADIENT MAPS ASSOCIATED WITH ACTIONS OF REAL REDUCTIVE GROUPS

Oluwagbenga Joshua Windare "Università di Parma"

Abstract. This talk is concerned with the Hamiltonian action of a complex reductive group $U^{\mathbb{C}}$ on a Kähler manifold (Z, ω). The momentum map associated with this action helps to analyze the geometric properties of the action. Our main goal is to investigate a refinement of this setting, namely, the action of a compatible subgroup G of $U^{\mathbb{C}}$ on a real submanifold X of Z. One can study the geometry of the G-action on X using the associated gradient map and its norm square function, using techniques from convex geometry, complex geometry, Lie group, and Morse theory.

We will discuss the classical Hilbert-Mumford criterion in Geometric Invariant Theory for the G-action on X. This gives an explicit numerical criterion for testing the (semi, poly)stability of a point in terms of an important G-equivariant function called the maximal weight function. It is a purely numerical condition, so it is very useful for practical reasons. We will also discuss the existence and uniqueness of the optimal destabilizing vector under different notions of stability. The optimal destabilizing vector is the basic tool used by Teleman and Bruasse for understanding the analog of the Harder-Narasimhan assignment in the finite-dimensional framework. This is a joint work with Leonardo Biliotti.

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