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The Ideal Theory and Arithmetic of Rings, Monoids, and Semigroups. Special Session A3

Scott Chapman
Sam Houston State University

Marco D'Anna
University of Catania, ITALY

During the last 20 years, the theory involving the structure of the arithmetic and ideal theory of various algebraic structures has been a popular topic and taken several important steps forward. Many applications of this theory, with particular attention to the multiplicative monoids of integral domains and their combinatorial or numerical applications to ring theory, have appeared throughout the mathematical literature. It is the aim of this session to review recent developments in this area by bringing together researchers from different areas of algebra under the umbrella of commutative monoids, semigroups, and rings. Topics to be covered include multiplicative ideal theory and general ideal systems, arithmetic in Krull and Prüfer monoids, commutative monoid rings, integer-valued polynomials, numerical monoids and congruence monoids, direct sum decompositions of modules, and various aspects of non-unique factorization.

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Schedule and Abstracts

July 23, 2024

11:00–11:20 Square-difference factor absorbing ideals of a commutative ring
Ayman Badawi* (The American University of Sharjah, Sharjah, UAE)
David F. Anderson (University of Tennessee, Knoxville, USA)
Jim Coykendall (Clemson University, South Carolina, USA)

Abstract. Let R be a commutative ring with $1 \neq 0$. A proper ideal I of R is a *square-difference factor absorbing ideal* (sdf-absorbing ideal) of R if whenever $a^2 - b^2 \in I$ for $0 \neq a, b \in R$, then $a + b \in I$ or $a - b \in I$. In this paper, we introduce and investigate sdf-absorbing ideals.

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11:30–11:50 Star operations related to Polynomial closure
Francesca Tartarone* (Università degli Studi Roma Tre, ITALY)
Dario Spirito (Università degli Studi di Udine)

Abstract. Let E be a subset of K. A polynomial $f(X) \in K[X]$ is *D-integer-valued* over E if $f(E) \subseteq D$, and the set $\mathrm{Int}(E, D)$ of such polynomials is a subring of $K[X]$. The *polynomial closure of* E *in* D is the largest subset F of K such that $\mathrm{Int}(E, D) = \mathrm{Int}(F, D)$ ([1]). Such a closure has been studied in several contexts from a topological point of view: for example, if D

is a valuation domain, the polynomial closure is a topology if and only if D has dimension 1 ([1, Theorem 5.3]) and [4, Theorem 2.7]); when D is also a rank-one discrete valuation domain (DVR) this topology coincides with the v-adic topology ([1, Proposition 4.5]). The polynomial closure can also be studied as a star-operation (see [2]). Genralizing results obtained in [3] for some classes of Prüfer domains, we show that the polynomial closure and the v-operation coincide if D is an integrally closed domains or if D has residue characteristic 0.

References

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12:00–12:20 Multiplicative lattices, their primes and spectral spaces

Carmelo Antonio Finocchiaro (University of Catania, ITALY)

Abstract. Prime ideals play a crucial role in multiplicative ideal theory, being a key tool for investigating ideal-theoretic properties of commutative rings. Moreover prime spectra are central in algebraic geometry, since they are foundation of scheme theory. However many other algebraic structures admit a prime spectrum: non-commutative rings, monoids, groups, etc. The aim of this talk is to introduce and study a natural framework where to study spectra, that of multiplicative lattices. We will focus on some ideas developed in [1] and [2].

References

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12:30–12:50 The isomorphism problem for ideal class monoids of numerical semigroups

Pedro A. García-Sánchez (University of Granada, SPAIN)

Abstract. A *numerical semigroup* S is a submonoid of $(\mathbb{N}, +)$ such that $\mathbb{N} \setminus S$ has finitely many elements, where $\mathbb{N}$ denotes the set of non-negative integers. The set $\mathbb{N} \setminus S$ is know as the set of gaps of S.

A set of integers I is a *(relative) ideal* of S if $I + S \subseteq I$ and I has a minimum (having a minimum is equivalent to $a + I \subseteq S$ for some integer a). On the set of ideals of S, we define the following relation: $I \sim J$ if there exists an integer z such that $I = z + J$. The set of ideals modulo this equivalence relation is known as the *ideal class monoid* of S, denoted $\mathcal{C}\ell(S)$. Addition of two classes $[I]$ and $[J]$ is defined in the natural way: $[I] + [J] = [I + J]$. The ideal classs monoid of a numerical semigroup was introduced in [1], where some basic properties and bounds for its cardinality where given.

There is a one to one correspondence between classes of ideals in $\mathcal{C}\ell(S)$ and normalized ideals of S, that is, ideals of S whose minimum is zero: in each class $[I]$ we choose $-\min(I) + I \in [I]$. Denote by $\mathfrak{I}_0(S)$ the set of normalized ideals of S, which is a submonoid of the set of relative ideals of S. The above correspondence is indeed a monoid isomorphism between $\mathcal{C}\ell(S)$ and $\mathfrak{I}_0(S)$. By using this correspondence, in [2], we gave new bounds for the cardinality of the ideal class monoid of a numerical semigroup with the help of Apéry sets and Kunz coordinates. We also studied several properties and notable elements of the posets $(\mathfrak{I}_0(S), \subseteq)$ and $(\mathfrak{I}_0(S), \preceq)$ (where $I \preceq J$ if there exists $K \in \mathfrak{I}_0(S)$ such that $I + K = J$) that reflected attributes and invariants of the semigroup S.

In this talk, we prove that if S and T are numerical semigroups such that there is a poset isomorphism from $(\mathfrak{I}_0(S), \subseteq)$ to $(\mathfrak{I}_0(T), \subseteq)$, then S and T must be equal. To achieve this result, we first prove that the poset of gaps of a numerical semigroup S, with respect to the order induced by S ($a \leq_S b$ if $b \in a+S$), completely determines S. The poset $(\mathbb{N} \setminus S, \leq_S)$ is isomorphic to the poset of ideals $\mathfrak{P}_0(S) = \{\{0, g\} + S : g \in \mathbb{N} \setminus S\}$ (under inclusion). We then show that any isomorphism from $(\mathfrak{I}_0(S), \subseteq)$ to $(\mathfrak{I}_0(T), \subseteq)$ restricts to an isomorphism between $(\mathfrak{P}_0(S), \subseteq)$ and $(\mathfrak{P}_0(T), \subseteq)$.

We will also prove that if S and T are numerical semigroups whose ideal class monoids are isomorphic, then S and T must be equal. To this end, we will use the fact that quarks in $(\mathfrak{I}_0(S), +)$ are unitary extensions of S (oversemigroups of S that differ in one element with S), and that a numerical semigroup is completely determined by its unitary extensions. Induction on the genus (number of gaps) of S and T, which is proven to be the same, is the last ingredient needed to finish the proof.

We will finish the talk with some questions regarding the poset $(\mathfrak{I}_0(S), \preceq)$.

References

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2:30–2:50 Norms and elasticities in rings of algebraic integers

Jim Coykendall* (Clemson University, USA)

Jared Kettinger (Clemson University, USA)

Abstract. One of the central notions in the study of factorization in monoids and domains is the notion of elasticity. If we let X denote a monoid or domain, and $x \in X$ can be factored as

$$x = \pi_1 \pi_2 \cdots \pi_n$$

with each $\pi_i \in X$ an irreducible, then we say that x has a factorization of length n. We now define the elasticity of x as

$$\rho(x) = \sup\{\frac{n}{m} | \text{where } x \text{ has (irreducible) factorizations of lengths } n \text{ and } m\}.$$

The elasticity of X can be obtained by taking the supremum over the elasticities of all of the atomic elements in X.

It is well-known (see [3] and [4]) that if R denotes a ring of algebraic integers, then the elasticity of R is given by

$$\rho(R) = \begin{cases} \frac{D(\mathrm{Cl}(R))}{2}, & \text{if } R \text{ is not a UFD} \\ 1, & \text{if } R \text{ is a UFD} \end{cases}$$

where $D(\mathrm{Cl}(R))$ is the Davenport constant of the class group of R.

If R is a ring of algebraic integers, the set of integral norms of R forms an atomic monoid that has been shown, in many cases, to effectively mirror the factorization structure of the parent ring R. For instance, if R is a ring of integers with quotient field F, and F is Galois over $\mathbb{Q}$, then R is a UFD if and only if $N(R)$, the set of integral norms of R, is a UFM (see [1]). An analogous result also holds for HFDs ([2]).

In this talk, we will discuss a concept that generalizes the notion of the Davenport constant for a finite abelian group. We then show that this is the "right" notion needed to talk about the relationship between the elasticity of the set of norms and its parent ring in the Galois case. Special attention will then be devoted to the quadratic case where some strong results may be obtained.

References

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3:00–3:20 The Ideal Theory and Arithmetic of Rings, Monoids, and Semigroups
Paul Pollack (University of Georgia, USA)

Abstract. A 1927 conjecture of Emil Artin predicts that for any integer g, not -1 and not a square, there are infinitely many primes p for which the multiplicative group mod p is generated by g. While Artin's conjecture remains unsolved, it has generated quite a lot of good mathematics. I will discuss applications of some of this theory to the elasticity of orders in quadratic fields. As one example, the Generalized Riemann Hypothesis implies that infinitely many orders inside $\mathbb{Q}(\sqrt{2})$ are half-factorial, confirming a conjecture of Coykendall.

3:30–3:50 Monoid Algebras and Weak Notions of ACCP
Felix Gotti (MIT, USA)

Abstract. For a submonoid M of a torsion-free abelian group and a commutative ring R, let $R[M]$ denote the monoid algebra of M over R. In this talk, we will discuss some recent progress on weaker notions of the ACCP property (i.e., every ascending chain of principal ideals stabilizes). In particular, we will discuss the ascent of such properties from the pair (M, R) to the monoid algebra $R[M]$. We will focus on the special case where R (and so $R[M]$) is an integral domain.

4:00–4:20 Algebraic Properties of Subsemigroups and Semigroup Ideals of Factorial Monoids
Paul Baginski (Fairfield University, USA)

Abstract. Consider a factorial monoid $F = F^\times \times \mathcal{F}(P)$, where P is a set of prime elements and $F^\times$ are the units of F. We will discuss the general algebraic and arithmetic properties of subsemigroups H of F, determining when they are Krull, root closed, bounded factorization, etc. Attention will be paid to subtle technical details that arise when adapting concepts from the factorization theory of monoids to the setting of semigroups. Additionally, the role of the units $F^\times$, particularly the set $F^\times \cap H$, will be discussed in detail. We then will specialize to semigroup ideals, namely nonempty subsets H of F satisfying $HF = H$. Here, many of the technical concerns involving units are quickly resolved and we obtain a much clearer picture of the algebra, which is often wild from a factorization-theoretic perspective. In particular, we will exhibit a large class of semigroup ideals which exhibit pathological factorization properties in the "furcus" family of conditions (bifurcus, m-furcus, and multifurcus).

5:00–5:20 Skew Mal'cev-Neumann Series Ring Over a Dedekind Domain
Daniel Z. Vitas* (University of Ljubljana, SLOVENIA)
Daniel Smertnig (University of Ljubljana, SLOVENIA)

Abstract. There are only a few known constructions of Dedekind prime rings. Our goal is to give a new one. Let D be a commutative Dedekind domain. We study the ring of skew Mal'cev-Neumann series,

$$R = D(\!(G; \alpha)\!),$$

for an ordered group G and a homomorphism $\alpha : G \to \mathrm{Aut}(D)$. We show that R is a (simple) noncommutative Dedekind domain under some assumptions on α. Furthermore, the canonical morphism from the ideal class group of D to the group of stable isomorphism classes of R-ideals is surjective. This gives us a way to construct new examples of noncommutative Dedekind domains with some control over their simple modules.

5:30–5:50 Egyptian integral domains and the ring of reciprocals

Lorenzo Guerrieri * (Jagiellonian University in Krakow, POLAND)
Neil Epstein (George Mason University, USA)
K. Alan Loper (Ohio State University, USA)

Abstract. An Egyptian fraction

$$q = \frac{1}{a_1} + \ldots + \frac{1}{a_n}$$

is a representation of a rational number q as a sum of distinct unit fractions. It is well-known since ancient time (the first known proofs goes back to Fibonacci) that any rational number q can be represented as an Egyptian fraction.

In this talk we start by discussing the existence of Egytpian fractions in arbitrary integral domains. We say that an element x of an integral domain D is *Egyptian* if there exist distinct elements $d_1, \ldots, d_n$ such that

$$x = \frac{1}{d_1} + \ldots + \frac{1}{d_n},$$

and that D is an *Egyptian domain* if all its elements are Egyptian. The ring of integers $\mathbb{Z}$ is clearly an Egyptian domain. We show that also the rings with nonzero Jacobson radical are Egyptian while polynomial rings over any ring are not Egyptian.

For this study, it is useful to consider the ring of reciprocals $R(D)$, defined as the subring of the quotient field of D generated by all fractions $\frac{1}{d}$ for nonzero $d \in D$. It can be observed that a domain D is Egyptian if and only if $R(D)$ coincides with the quotient field of D.

In the second part of the talk we focus on the description of the ring $R(D)$ in the case D is a polynomial ring over a field in a finite number of variables.

6:00–6:20 Asymptotic behaviour of the v-number of homogeneous ideals

Antonino Ficarra (University of Évora, PORTUGAL)

Abstract. Let $S = K[x_1, \ldots, x_n]$ be the standard graded polynomial with coefficients over a field K, and let $I \subset S$ be a homogeneous ideal. The v-number of I is defined as the minimum degree of an homogeneous polynomial $f \in S$ such that $(I : f) \in \mathrm{Ass}(I)$ is an associated prime of I. This invariant was introduced in relation to minimum distance functions and Reed-Muller type codes. In the present talk, we show the following

Theorem 1. [3, Theorem 1.1], [5, Theorems 3.1 and 4.1] *Let $I \subset S$ be a graded ideal. Then, for all $k \gg 0$, $\mathrm{v}(I^k)$ is a linear function of the form $\alpha(I)k + b$ where*

$$\lim_{k \to \infty} \frac{v(I^k)}{k} \; = \; \alpha(I)$$

is the initial degree of I and $b \in \mathbb{Z}$. Here $\alpha(I)$ is the minimum degree of an homogeneous element belonging to I.

We then survey the recent numerous studies on this and related topics, like monomial ideals, integer programming and graded filtrations, and then we discuss some open questions.

References

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July 24, 2024

11:30–11:50 Differences in sets of lengths for monoids of plus-minus weighted zero-sum sequences

Wolfgang A. Schmid * (University of Paris 8, FRANCE)
Kamil Merito (University of Paris 8, FRANCE)
Oscar Ordaz (Universidad Central de Venezuela, VENEZUELA)

Abstract. Let $(G, +)$ be a finite abelian group. A (finite) sequence of elements $g_1 \dots g_k$ is called a zero-sum sequence if $g_1 + \cdots + g_k = 0$; in the current context sequences that just differ in the ordering of the terms are considered as equal. The set of all zero-sum sequences over G, denoted $\mathcal{B}(G)$, forms a submonoind of the monoid of all sequences over G, denoted $\mathcal{F}(G)$. It is a Krull monoid and there are numerous results on its arithmetic, in particular as results on its arithmetic translate directly into results on the arithmetic of rings of algebraic integers.

More recently the following related monoid has been investigated (see the references). A sequence is called a plus-minus weighted zero-sum sequences if $\epsilon_1 g_1 + \cdots + \epsilon_k g_k = 0$ for some choice of 'weights' $\epsilon_i \in \{+1, -1\}$. The set of all these sequences is denote $\mathcal{B}_{\pm}(G)$; it is also a submonoid of $\mathcal{F}(G)$. For most G, it is not a Krull monoid yet still a C-monoid. Moreover, its arithmetic is related to the arithmetic of normsets for rings of integers of quadratic number fields.

When investigating the arithmetic of monoids, sets of lengths are an important notion. For an element a of a multiplicative monoid H, one says that a has a factorization of lengths l if there are irreducible elements $a_1, \dots, a_l$ such that $a = a_1 \dots a_l$. One denotes by $\mathsf{L}(a)$ the set of all l that are a lengths of a.

Since these monoids, $\mathcal{B}_{\pm}(G)$, are finitely generated monoids the Structure Theorem of Sets of Lengths holds (see for example [1, 4] for details), that is, all its sets of lengths are Almost Arithmetical Multiprogressions with a global bound M and a difference from a certain finite set of differences $\Delta^*(\mathcal{B}_{\pm}(G))$.

The goal of this talk is to present results on the sets $\Delta^*(\mathcal{B}_{\pm}(G))$, in particular its maximum. In case the order of G is odd, we obtain fairly precise results, in particular the maximum is equal to $\exp(G) - 2$. For groups of even order, the situation is much less clear and $\Delta^*(\mathcal{B}_{\pm}(G))$ can contain considerably larger elements.

References

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12:00–12:20 On an Inverse Zero-sum Question for Rank Two Groups

David J. Grynkiewicz * (University of Memphis, USA)
John Ebert (University of Memphis, USA)

Abstract. For an abelian group G, the Davenport Constant of G is the smallest integer ℓ such that any sequence of ℓ terms from G must contain a non-trivial zero-sum subsequence. If $k \geq 0$ is an integer, one can refine this question by asking how long a sequence of terms from G must be to guarantee a nontrivial zero-sum of length at most $\mathsf{D}(G)-k$, so with k less terms than that guaranteed by definition of the Davenport constant. Let $\mathsf{s}_{\leq \mathsf{D}(G)-k}(G)$ denote this minimal ℓ such that any sequence of ℓ terms from G must contain a nontrivial zero-sum subsequence of length at most $\mathsf{D}(G)-k$. Wang and Zhao determined its precise value for rank two abelian groups $G = C_m \oplus C_n$ with $m \mid n$, showing

$$\mathsf{s}_{\leq \mathsf{D}(G)-k}(G) = m+n-1+k \quad \text{for } k \in [0, m-1]$$

(the constant is infinite for larger values of k). At the extremal values $k=0$ and $k=m$, this recovers well-known exact formulas for the Davenport Constant and the η Constant of rank two abelian groups (the latter asking for a nontrivial zero-sum subsequence of length at most the exponent of G). It remained an open question to characterize the structure of all extremal sequences of length $m+n-2+k$ failing to contain a nontrivial zero-sum of length at most $\mathsf{D}(G)-k = m+n-1-k$. Prior work completely reduced the general characterization problem to that for rank two p-groups of the form $C_p \oplus C_p$, with partial progress resolving the case of small $k \leq \frac{2p+1}{3}$. Here, we resolve all open cases, showing that a sequence of $2p-2+k$ terms avoiding any nontrivial zero-sum with length at most $2p-1-k$, where $k \in [2, p-2]$, must consist of three distinct terms e_1, e_2 and e_1+e_2, with e_1 and e_2 repeated $p-1$ times and e_1+e_2 repeated k times, for some basis (e_1, e_2) of G. Hence all extremal sequences have the form $e_1^{p-1} \cdot e_2^{p-1} \cdot (e_1+e_2)^k$.

12:30–12:50 On the arithmetic of the monoid of plus-minus weighted zero-sum sequences

Andreas Reinhart * (University of Graz, AUSTRIA)
Florin Fabsits (University of Graz, AUSTRIA)
Alfred Geroldinger (University of Graz, AUSTRIA)
Qinghai Zhong (University of Graz, AUSTRIA)

Abstract. Let G and G_1 be abelian groups and let $\mathcal{F}(G)$ be the free abelian monoid with basis G. By $\mathcal{B}_{\pm}(G) = \{\prod_{i=1}^n g_i \mid n \in \mathbb{N}_0, (g_i)_{i=1}^n \in G^n \text{ and } \sum_{i=1}^n \varepsilon_i g_i = 0 \text{ for some } (\varepsilon_i)_{i=1}^n \in \{-1,1\}^n\} \subseteq \mathcal{F}(G)$, we denote the monoid of plus-minus weighted zero-sum sequences.

In this talk, we discuss and describe when $\mathcal{B}_{\pm}(G)$ satisfies various well-known properties, like being a Mori monoid, a C-monoid or a finitely generated monoid. As a byproduct, we rediscover and strengthen some of the results of [1] and [2].

We will also put our focus on two problems that were profoundly studied for many other types of monoids. These problems are the isomorphism problem and the characterization problem.

- **The isomorphism problem:** *Let $\mathcal{B}_{\pm}(G)$ and $\mathcal{B}_{\pm}(G_1)$ be isomorphic (as monoids). Determine when G and G_1 are isomorphic (as groups).*
- **The characterization problem:** *Suppose that the systems of sets of lengths of $\mathcal{B}_{\pm}(G)$ and $\mathcal{B}_{\pm}(G_1)$ coincide. Describe when G and G_1 are isomorphic (as groups).*

The main results of our talk provide partial solutions to each of these problems.

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[2] A. Geroldinger, F. Halter-Koch, and Q. Zhong, *On monoids of weighted zero-sum sequences and applications to norm monoids in Galois number fields and binary quadratic forms*, Acta Math. Hungarica **168** (2022), 144–185.

2:30–2:50 The stable rank of a ring of integer-valued polynomials

Sophie Frisch (Graz University of Technology, AUSTRIA)

Abstract. The stable rank of a ring, introduced by Bass and simplified by Vaserstein, is an important invariant in algebraic K-theory, but, we think, underappreciated in commutative ring theory.

Stable rank is comparable to Krull dimension: By a result of Heitmann $s.r.(R) \leq \dim(R) + 1$, for Prüfer rings. (But s.r. is not really a measure of dimension, since $s.r.(R) = 1$ for every local ring.)

We will show $s.r.(R) = 2$ for some 2-dimensional Prüfer rings, namely, rings of integer-valued polynomials over rings of integers in number fields (other than imaginary quadratic) and discuss possible generalizations.

References

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3:00–3:20 Nontriviality of rings of integral-valued polynomials

Giulio Peruginelli* (Università di Padova, ITALY)
Nicholas J. Werner (State Univ. of New York at Old Westbury, USA)

Abstract. In this work we consider rings of integral-valued polynomials over subsets S of the ring of all algebraic integers $\overline{\mathbb{Z}}$, defined as $\mathrm{Int}_{\mathbb{Q}}(S, \overline{\mathbb{Z}}) = \{f \in \mathbb{Q}[X] \mid f(S) \subseteq \overline{\mathbb{Z}}\}$. The family of these rings has been introduced by Alan Loper and Nicholas J. Werner in 2012 to obtain an example of a Prüfer domain strictly contained between $\mathbb{Z}[X]$ and the classical ring of integer-valued polynomials $\mathrm{Int}(\mathbb{Z})$. We correct here a minor wrong claim of that paper, namely, that for any $S \subseteq \overline{\mathbb{Z}}$, the ring $\mathrm{Int}_{\mathbb{Q}}(S, \overline{\mathbb{Z}})$ is always nontrivial, i.e., it strictly contains $\mathbb{Z}[X]$. For example, if S comprises all the roots of unity $\xi_n, n \in \mathbb{N}$ it is not difficult to show that $\mathrm{Int}_{\mathbb{Q}}(S, \overline{\mathbb{Z}}) = \mathbb{Z}[X]$ using the fact that the ring of integers of $\mathbb{Q}(\xi_n)$ is monogenic. We completely characterize those subsets S of $\overline{\mathbb{Z}}$ for which $\mathrm{Int}_{\mathbb{Q}}(S, \overline{\mathbb{Z}})$ is nontrivial in terms of pseudo-divergent sequences and pseudo-stationary sequences contained in S with respect to some fixed extension of the p-adic valuation to $\overline{\mathbb{Q}}$, as p runs through the set of prime integers. These sequences have been introduced by Chabert in 2010 in order to study the polynomial closure of subsets of rank one valuation domains.

We produce several examples of subsets of $\overline{\mathbb{Z}}$ of unbounded degree for which the corresponding ring $\mathrm{Int}_{\mathbb{Q}}(S, \overline{\mathbb{Z}})$ is trivial or not. In particular, we show that the monogenicity of the ring of integers of $\mathbb{Q}(s)$, for $s \in S$, is not a necessary condition for $\mathrm{Int}_{\mathbb{Q}}(S, \overline{\mathbb{Z}})$ to be trivial.

References

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3:30–3:50 Integer-valued polynomials over subrings of matrix algebras

Valentin Havlovec (Graz University of Technology, AUSTRIA)

Abstract. Integer-valued polynomials with coefficients in noncommutative algebras have been studied since around 2010. As there is no substitution homomorphism in this case, it is not clear when such polynomials form a ring. In 2012, N. Werner gave a positive answer for integer-valued polynomials with matrix coefficients, by showing that $\mathrm{Int}(\mathrm{M}_n(D)) = \{f \in \mathrm{M}_n(K)[x] \mid f(\mathrm{M}_n(D)) \subseteq \mathrm{M}_n(D)\}$, where D is an integral domain with quotient field K, is a subring of $\mathrm{M}_n(K)[x]$. S. Frisch showed that this also holds for integer-valued polynomials over upper triangular matrices. These results were extended by J. Sedighi Hafshejani, A. R. Naghipour, and M. R. Rismanchian to certain kinds of block matrix algebras. We study subalgebras of $\mathrm{M}_n(D)$ of the following form: let $\precsim$ be a preorder on $\{1,\ldots,n\}$ and define $\mathrm{M}_{\precsim}(D) = \{(a_{ij}) \in \mathrm{M}_n(D) \mid a_{ij} \neq 0 \Rightarrow i \precsim j\}$. In other words, the ring $\mathrm{M}_{\precsim}(D)$ consists of those matrices whose coefficients in certain positions (determined by $\precsim$) are zero, while the others range freely over D. We show that the set of integer-valued polynomials $\mathrm{Int}(\mathrm{M}_{\precsim}(D)) = \{f \in \mathrm{M}_{\precsim}(K)[x] \mid f(\mathrm{M}_{\precsim}(D)) \subseteq \mathrm{M}_{\precsim}(D)\}$ is a ring and give a description in terms of polynomials with coefficients in K.

References

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4:00–4:20 Integer-Valued Polynomials for Semidomains

Harold Polo* (University of California at Irvine, USA)
Scott Chapman (Sam Houston State University, USA)
Nathan Kaplan (University of California at Irvine, USA)

Abstract. A semidomain is a subsemiring of an integral domain. One might think of a semidomain as an integral domain in which the condition of having additive inverses is relaxed. In our discussion, we generalize the notion of integer-valued polynomials for the class of semidomains. This talk is based on joint works with Scott T. Chapman and Nathan Kaplan.

5:00–5:20 Bounds for syzigies of monomial curves

Alessio Moscariello* (University of Catania, ITALY)
Giulio Caviglia (Purdue University, USA)
Alessio Sammartano (Politecnico di Milano, ITALY)

Abstract. Despite the simplicity of the objects involved, this question is still not well understood. A numerical monoid $\Gamma \subseteq \mathbb{N}$ is a cofinite additive submonoid of $\mathbb{N}$. Every numerical monoid has a unique minimal set of generators $g_0 < g_1 < \cdots < g_e \in \Gamma$, and we write $\Gamma = \langle g_0, g_1, \ldots, g_e\rangle$ to denote this fact. The cardinality $e+1$ of the minimal set of generators is called the embedding dimension of Γ, and it is denoted by $edim(\Gamma)$. The monoid morphism $\varphi : \mathbb{N}^{e+1} \to \mathbb{N}$ defined by $\varphi(a_0, a_1, \ldots, a_e) = \sum_{i=0}^{e} a_i g_i$ yields an isomorphism $\Gamma \cong \mathbb{N}^{e+1}/\ker(\varphi)$, where $\ker(\varphi) = \{(\mathbf{a}, \mathbf{b}) \in \mathbb{N}^{e+1} \times \mathbb{N}^{e+1} \mid \varphi(\mathbf{a}) = \varphi(\mathbf{b})\}$ is the kernel congruence of φ. A *minimal presentation* of Γ is a minimal set of generators of the congruence $\ker(\varphi)$. Minimal presentations are not unique, but their cardinality depends only on Γ. This cardinality is denoted by $\rho(\Gamma)$, and it is referred to as the number of minimal relations of Γ.

The main question we consider in this talk is: how many minimal relations can a numerical monoid have? The picture for lower bounds is quite clear, since it is known that $\rho(\Gamma) \geq e$, with equality attained for *complete intersection* semigroups. On the other hand, finding sharp upper bounds for the number of minimal relations of a numerical monoid is a surprising complex problem.

Let K be a field and $P = K[[x_0, x_1, \ldots, x_e]]$.

The semigroup ring of Γ is $R_\Gamma = K[[t^\gamma : \gamma \in \Gamma]] = K[[t^m, t^{g_1}, \ldots, t^{g_\nu}]] \subseteq \Bbbk[[t]]$.

We have $R_\Gamma = P/I_\Gamma$, where I_Γ is the toric ideal of the semigroup Γ

$$I_\Gamma = \left(x_0^{\alpha_0} \cdots x_\nu^{\alpha_e} - x_0^{\beta_0} \cdots x_\nu^{\beta_e} \mid \sum_{i=0}^{\nu} \alpha_i g_i = \sum_{i=0}^{\nu} \beta_i g_i \right).$$

The number of minimal relations $\rho(\Gamma)$ is actually the number of minimal generators of I_Γ, and thus it is possible to bound the number of relations in this way by using techniques coming from commutative algebra.

In this talk, we provide an useful connection between the world of semigroup rings and toric ideals and that of monomial ideals, by associating to I_Γ a monomial ideal J_Γ having larger Betti numbers; using this approach, we can thus bound $\rho(\Gamma)$ by translating the original problem in the context of monomial ideals, granting us access to some classical results in this context. We provide a general bound for the number of relations, and for the Betti numbers of I_Γ at large, in function of e and g_0, as well as finding some cases in which this bound is sharp. Furthermore, we use this approach to study a question of Herzog and Stamate, asking for a bound for $\rho(\Gamma)$ in function of $g_e - g_0$.

References

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5:30–5:50 On factorization invariants of ideal extensions of free commutative monoids

Carmelo Cisto* (University of Messina, ITALY)
Pedro A. García Sánchez (University of Granada, SPAIN)
David Llena (University of Almeria, SPAIN)

Abstract. Let $\mathbb{N}$ be the set of non negative integers and I be a non-empty subset of $\mathbb{N}$. We denote by $\mathbb{N}^{(I)}$ the additive monoid consisting of all sequences $\mathbf{n} = (n_i)_{i \in I}$ such that $n_i = 0$ but for finitely many $i \in I$. If S is a submonoid of $\mathbb{N}^{(I)}$, the set of gaps of S is defined as $\mathcal{H}(S) = \mathbb{N}^{(I)} \setminus S$. An element $\mathbf{a}$ of S is an *atom* if whenever $\mathbf{a} = \mathbf{b} + \mathbf{c}$ for some $\mathbf{b}, \mathbf{c} \in S$, then either $\mathbf{b} = \mathbf{0}$ or $\mathbf{c} = \mathbf{0}$. The set of atoms of S is denoted by $\mathcal{A}(S)$. S is called an *ideal extension* of $\mathbb{N}^{(I)}$, if $S^* = S \setminus \{\mathbf{0}\}$ is an ideal of $\mathbb{N}^{(I)}$, that is, $S^* + \mathbb{N}^{(I)} \subseteq S^*$. Moreover, for a submonoid S of $\mathbb{N}^{(I)}$, we say that S is a *gap absorbing monoid* if

(1) $2\mathcal{H}(S) \subseteq \mathcal{H}(S) \cup \mathcal{A}(S) \cup 2\mathcal{A}(S)$, and
(2) $\mathcal{H}(S) + \mathcal{A}(S) \subseteq \mathcal{A}(S) \cup 2\mathcal{A}(S)$.

In this work, we study some invariants of Factorization Theory of these kind of monoids. In particular, we provide some properties about Betti Elements, Delta set, Catenary degree and Omega primality (see [2] and [3] for the definition of these invariants). For instance, for the Delta set $\Delta(S)$, a conjecture by N. Baeth in [1,Conjecture 4.16] states that $\Delta(S) = \{1\}$ holds for every complement finite ideal S in $\mathbb{N}^d$, with d any positive integer. We provide an affirmative answer to this question in the case $S \subseteq \mathbb{N}^{(I)}$ is a gap absorbing monoid. This work is a continuation of [1], and is dedicated to its author.

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6:00–6:20 Discrete valuation overrings of a Noetherian ring

Gyu Whan Chang (Incheon National University, SOUTH KOREA)

Abstract. Let R be a Noetherian ring and P be a regular prime ideal of R. Then there is a rank one discrete valuation overring of R with (regular) maximal ideal M, so that $M \cap R = P$. This is a natural generalization of Chevalley's result that every local Noetherian domain is dominated by a rank one discrete valuation overring.

E–mail: `scott.chapman@shsu.edu`.