AMS-UMI International
Joint Meeting 2024
Palermo, July 23–26, 2024

Approximation Theory and Application
Special Session B10

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This special session is focused on numerical and analytical aspects as well as applications of multivariate approximation problems. Areas of our speakers will discuss kernel-based methods such as applications in high-dimensional interpolation, quasi-interpolation or regression settings. The problems will come from learning-based methods such as deep neural networks or physics-informed neural networks, from signal or image-based analysis settings, as well as computations of partial differential or fractional differential problems. Finally, things might live in cloud-based surfaces, high-dimensional spheres, manifolds, as well as set-valued and vector-valued domains, and our techniques may use analytical, statistical or Bayesian approaches.

For more information visit `https://umi.dm.unibo.it/jm-umi-ams/special-sessions/`.

Schedule and Abstracts

July 25, 2024

11:30–11:50 Variably Scaled Persistence Kernels (VSPKs) for persistent homology applications

Stefano De Marchi (University of Padova, ITALY)

Abstract. In recent years, various kernels have been proposed in the context of *persistent homology* to deal with *persistence diagrams* in supervised learning approaches. In this paper, we consider the idea of variably scaled kernels, for approximating functions and data, and we interpret it in the framework of persistent homology. We call them *Variably Scaled Persistence Kernels (VSPKs)*. These new kernels are then tested in different classification experiments. The obtained results show that they can improve the performance and the efficiency of existing standard kernels.

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12:00–12:20 Products of Matrices and Cascade Networks

Mike Neamtu (Vanderbilt University, USA)

Abstract. For positive integers $L, N_0, N_1, ..., N_L$, we consider a collection of two-valued matrix functions $\{A_\ell\}_{\ell=1}^L$ on $[0, 1]$, called here a *cascade network* of depth L, given by

$$A_\ell(x) := \begin{cases} A_\ell^0, & x \in [0, \frac{1}{2}), \\ A_\ell^1, & x \in [\frac{1}{2}, 1]), \end{cases}$$

where $A_\ell^0, A_\ell^1 \in \mathbb{R}^{N_\ell \times N_{\ell-1}}$, $\ell = 1, \ldots, L$. One can use these matrices to generate matrix-valued functions recursively by

$$y_\ell(x) := A_\ell y_{\ell-1}(\alpha(x)), \quad x \in [0,1], \quad \ell = 1, \ldots, L,$$

where

$$\alpha(x) := \begin{cases} 2x, & x \in [0, \frac{1}{2}), \\ 2x - 1, & x \in [\frac{1}{2}, 1]), \end{cases}$$

and where y_0 is a given input matrix-valued function on $[0,1]$. Cascade networks are closely related to so-called subdivision algorithms and refinement equations of multi-resolution analysis. For example, vector-valued stationary and non-stationary subdivision schemes can be expressed in terms of recursions of the above form, which give rise to products of matrices. In this talk, we will address the problem of under what conditions do these products converge to a continuous limit as $L \to \infty$.

12:30–12:50 On the Convergence of Multiscale Kernel Regression under Minimalistic Assumptions

Armin Iske (Universität Hamburg, GERMANY)

Abstract. We analyse the convergence of kernel regression under minimalistic assumptions on the data and on the kernel. To this end, we prove error estimates and convergence rates with respect to the kernel's native space norm. Our results are then transferred to multiscale kernel regression.

14:30–14:50 Provable approximations of multivariate functions on smooth manifolds using deep ReLU neural networks

Demetrio Labate (University of Houston, USA)

Abstract. The expressive power of deep neural networks is manifested by their remarkable ability to approximate multivariate functions in a way that appears to overcome the curse of dimensionality. This ability is exemplified by their success in solving high-dimensional problems where traditional numerical solvers fail due to their limitations in accurately representing high-dimensional structures. To provide a theoretical framework for explaining this phenomenon, we analyze the approximation of Hölder functions defined on a d-dimensional smooth manifold M embedded in R^D, with $d \ll D$, using deep neural networks. Here neural networks are identified with a class of structured parametric functions, consistently with the recent literature. We prove new uniform convergence estimates of the approximation and generalization errors by deep neural networks with ReLU activation functions, showing that such estimates do not depend on the ambient dimension D of the function but only on the lower manifold dimension d, in a precise sense. This result improves existing estimates established in the literature in a similar setting, where approximation and generalization errors were shown to depend weakly on the ambient dimension D. The result presented in this talk was recently published in [1].

References

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15:00–15:20 Kernel-Based Neural Operators

Matthew Lowery (University of Utah, USA)

Abstract. Operator learning is the task of learning a map from one function space to another. Recently, deep neural networks have been leveraged for this task, resulting in architectures like the Fourier Neural operator (FNO) and the Deep Operator Network (DeepONet). In this talk, we present two kernel-based neural operator architectures. First, we present the Kernel Neural Operator (KNO), an alternative to the FNO that uses one-to-two orders of magnitude fewer trainable parameters to achieve state-of-the-art accuracy on a range of operator learning tasks. Next, we present Ensemble DeepONets, which are kernel-enhanced, locality-aware ensemble operator models that leverage kernel-based partition-of-unity methods to improve the generalization capabilites of DeepONets. While kernels have found their uses in machine learning for over half a century, we believe our new methods represent a novel way of applying modern kernel approximation techniques to machine learning tasks.

15:30–15:50 Hyperparameter selection in adaptive kernel-based partition of unity methods by univariate global optimization tools

Roberto Cavoretto (University of Turin, ITALY)

Abstract. In this talk, in order to detect the kernel shape parameter and the subdomain size utilized in radial kernel-based partition of unity interpolation [4], we have considered univariate global optimization techniques [3]. Particularly, we induced optimistic and pessimistic improvement in the efficient global optimization method [1], and combined it with a leave-one-out cross validation (LOOCV) scheme [2] for each partitioned subdomain. Numerical results show the enhanced performance of the new technique compared to a basic LOOCV scheme.

References

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16:00–16:20 Some new perpectives in Korovkin-type approximation theory

Michele Campiti (University of Salento, ITALY)

Abstract. In the last decades, many aspects of Korovkin approximation theory have been deepened and nowadays we have a rather complete theory. A fairly comprehensive treatment of this theory and its main application can be found in [3, 4].

However, some questions have not yet been completely addressed and could increase further the interest toward this theory.

One of the problems which can be surely considered in this framework is the possibility of extending the Korovkin approximation theory in spaces of integrable functions or, in any case, in spaces of not necessarily continuous functions.

Of course, many papers have been devoted to this possibility and we have also the general universal Korovkin-type property (see [3, Theorem 3.2.1]) which allows to consider operators whose ranges are contained in a suitable Banach lattice.

Surely, among the most interesting applications of Korovkin-type approximation theory, we have to consider the convergence of sequences of operators associated with a positive projections (see [3, Chapter 6] and [1, 6, 7]).

In an abstract setting these applications are all framed in spaces of continuous functions and, as far as we know, we find no complete extension to more general spaces.

Some recent contribution in this direction can be found in [2]. The main idea used in [2] is to extend Radon measures to the space of bounded Borel measurable functions. In this way, it is possible to extend Bernstein-Schnabl operators to bounded Borel measurable functions. However, due to some assumptions used in the paper, this kind of extension does not cover some classical sequences of operators, such as Kantorovich operators.

Another attempt to extend to larger spaces the sequences of operators associated with a positive projection can be found in [5]. In this case, the extension covers the case of Kantorovich operators but only in the interval $[0, 1]$.

In this talk the authors present some further possibilities of considering sequences of positive operators associated with a positive projection in spaces of integrable functions and, in the

meantime, to include classical operators such as multi-dimensional Kantorovich or Bernstein-Durrmeyer operators in this extension.

References

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17:00–17:20 Variation, approximation, sampling-type operators
Laura Angeloni (University of Perugia, ITALY)

Abstract. In this talk we will discuss about some approximation results within the spaces of functions of bounded variation by means of sampling-type operators. BV-spaces furnish, indeed, an interesting setting for approximation processes, also suitable to model, from a mathematical point of view, certain problems related to Signal and Image Processing. In this direction, we will present estimates and convergence in variation results focusing the attention on the multidimensional case by means of the classical generalized sampling series, as well as their Kantorovich version, using the concept of variation in the sense of Tonelli. Indeed the geometrical aspects connected to the definition of the Tonelli variation become it suitable to address such kind of problems, also in view of the applicative connections. Some recent results about the nonlinear case will be also presented.

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17:30–17:50 Some approximation results and estimates for linear operators in Orlicz spaces

Luca Zampogni (University of Perugia, ITALY)

Abstract. We state and discuss some approximation results and give qualitative estimate on the rate of convergence for a general class of linear operators

$$T_w f(z) = \int_H \chi_w(z - h_w(t)) \cdot L_{h_w(t)} f d\mu_H(t),$$

where H, G are locally compact topological groups, $(\chi_w)_w$ is a family of kernels, $(h_w(\cdot))_w$ is a family of homeomorphisms from H to $h_w(H) \subset G$, and $(L_{h_w(t)})_{t,w}$ are linear operators from $M(G) = \{f : G \to \mathbb{R} \mid f \text{ is measurable}\}$ to $\mathbb{R}$. Particular choices of H, G and of the operators involved give rise to classical both discrete and integral operators, like the generalized sampling series, the Kantorovich sampling series, the Durrmeyer operators, discrete and integral convolutions and Mellin operators, in both the one and the multidimensional cases, allowing to introduce a general setting where to study both the convergence and the rate of convergence. We discuss some results concerning the convergence of the operators $T_w f$ to the function f in $C(G)$ (the space of uniformly continuous and bounded functions defined in G) and in Orlicz spaces $L^\varphi(G)$, and give also qualitative estimates for the rate of the convergence, both in $C(G)$ and in $L^\varphi(G)$. These estimates require the introduction of suitable Lipschitz classes and the definition of an appropriate modulus, due to the lack of a metric structure on the topological group G.

References

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18:00–18:20 On a sequence of positive linear operators on noncompact real intervals

Vita Leonessa (University of Basilicata, ITALY)

Abstract. The object of this talk is a sequence of positive linear operators of the form

$$C_n(f)(x) := \int_J \int_J \cdots \int_J f\left(\frac{x_1 + \ldots + x_n + r x_{n+1}}{n+r}\right) d\mu_x(x_1) \ldots d\mu_x(x_n) d\mu_n(x_{n+1}),$$

defined for an arbitrary interval J, $x \in J$, $n \geq 1$, $r \geq 0$, probability Borel measures μ_x and μ_n on J and a continuous real-valued function f on J with at most quadratic growth. Note that if J is a compact interval, then such operators turn into the generalized Kantorovich operators studied in [1]. Furthermore, for $r = 0$, they become the Bernstein-Schnabl operators on noncompact real intervals already studied in [3]. Approximation properties in weighted function spaces of continuous functions on J with respect to wide classes of weights have been investigated, establishing pointwise estimates for uniformly continuous bounded functions as well as with respect to weighted norms. Moreover, a weighted asymptotic formula has also been obtained. The latter result could possibly be used in studying some classes of evolution equations on J as done in [3]. All results presented are taken from [2].

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July 26, 2024

11:30–11:50 Vector-Valued Gaussian Processes for Approximating Divergence- or Rotation-free Vector Fields

Holger Wendland (University of Bayreuth, GERMANY)

Abstract. Gaussian processes are established tools for approximating unknown functions under the influence of uncertainties. The approximation of Gaussian processes of scalar-valued functions in the so-called regression setting have extensively been studied and are by now well understood. Less is known about vector-valued Gaussian processes. In this talk, I will discuss vector-valued Gaussian processes for the approximation of divergence-free functions. I will introduce the theory behind such Gaussian processes, link the theory to multivariate approximation theory and give error estimates for the predictive mean in various situations.

12:00–12:20 The Occupation Kernel Method for Learning Vector Fields with Constraints

Nicholas Fisher (Portland State University, USA)

Abstract. The occupation kernel method (OCK) has proven itself as a robust and efficient method for learning nonparametric systems of ordinary differential equations from trajectories in arbitrary dimensions. Using an implicit formulation provided by vector-valued reproducing kernel Hilbert spaces, we aim to show how the OCK method can be adapted to learn vector fields satisfying various physical constraints. In particular, by choosing an appropriate kernel, we can ensure that the learned vector fields analytically satisfy either solenoidal (divergence-free) and irrotational (curl-free) properties. We validate the proposed method through experiments on a variety of simulated and real datasets. It is shown that the added constraints often lead to better approximations in these application specific problems.

References

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12:30–12:50 The shape of the flat radial basis function interpolation limit

Elisabeth Larsson (Uppsala University, SWEDEN)

Abstract. Infinitely smooth radial basis functions (RBFs) $\phi(r)$, where $r = \|x - y\|$ for $x, y \in \mathbb{R}^d$, are in general equipped with a shape parameter ε such that $\phi_\varepsilon(r) = \phi(\varepsilon r)$ is used in approximations. The shape parameter has a significant effect on the accuracy [7] and different methods have been proposed to find the best shape parameter for a given problem [3]. In [1], it was shown that interpolation in the univariate flat RBF limit for distinct node points becomes the Lagrange interpolation polynomial. Some years later it was further shown that for unisolvent node sets, the multivariate flat RBF limit is the unique minimal degree multivariate polynomial interpolant [6, 9, 8]. The form of the interpolant $s_\varepsilon(x)$ was shown to be

$$s_\varepsilon(x) = P_K(x) + \varepsilon^2 P_{K+1}(x) + \varepsilon^4 P_{K+2}(x) + \cdots , \tag{1}$$

where K is the degree of the (multivariate) limit polynomial interpolant, and P_{K+j} is a polynomial of degree $K + 2j$. A recursive algorithm to compute these polynomials was provided in [6], but it can only be used stably for quite small shape parameters and for small numbers of points, which is the same as low values of K. A number of stable evaluation methods have later been derived, such as the so called RBF-QR method class [5, 4, 2], where the linearly dependent flat RBFs are replaced by a more well-conditioned basis. Here we go back to the first recursive formulation, and derive a corresponding non-recursive form that allows us to make some conclusions concerning which properties of the data and the underlying function that determine the best shape parameter. Based on those insights, we propose algorithms that can be used to find function approximations that may be more accurate than polynomial interpolation.

References

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14:30–14:50 A new framework for numerical integration
Grady B. Wright (Boise State University, USA)

Abstract. Numerical integration, or quadrature, is ubiquitous in mathematics, statistics, science, and engineering, with a history dating back to the ancient Babylonians. A standard approach to generating quadrature formulas is to pick a "nice" vector space of functions for which the formulas are exact, such as algebraic or trigonometric polynomials. For integration over intervals, this approach gives rise to Newton-Cotes and Gaussian quadrature rules. However, for geometrically complex domains in higher dimensions, this exactness approach can be challenging, if not impossible since it requires being able to exactly integrate basis functions for the vector space over the domains (or some collection of subdomains). Another challenge with determining good quadrature formulas arises when the integrand is not given everywhere over the domain, but only as samples at predefined, possibly "scattered" points (i.e., a point cloud), which is common in applications involving experimental measurements or when quadrature is a secondary operation to some larger computational effort. In this talk we introduce a new framework for generating quadrature formulas that bypasses these challenges. The framework only relies on numerical approximations of certain Laplace operators and on linear algebra. We show how several classic univariate quadrature formulas can arise from this framework and demonstrate its applicability to generating accurate quadrature formulas for geometrically complex domains (including surfaces) discretized with point clouds.

15:00–15:20 A Nyström method for solving fractional relaxation-oscillation equation
Maria Carmela De Bonis (University of Basilicata, ITALY)

Abstract. The relaxation-oscillator equation is the primary equation for describing the behaviour of physical phenomena that return to equilibrium after being disturbed. Fractional derivatives have been employed in order to represent slow relaxation and damped oscillation, obtaining the following fractional relaxation-oscillator equation

$$(D^\alpha y)(t) + \lambda y(t) = f(t), \quad t > 0, \tag{1}$$

where

$$(D^\alpha y)(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-x)^{m-\alpha-1} y^{(m)}(t) dt \tag{2}$$

is the Caputo's fractional derivative of order α with $m-1 < \alpha < m$ and $m = 1, 2$, $\lambda = \omega_0^2$ with ω_0 the natural frequency of the oscillator and f is an external force. For $0 < \alpha < 1$, (1) under the initial condition

$$y(0) = y_0$$

describes the relaxation with the power law attenuation; for $1 < \alpha < 2$, (1) under the initial conditions

$$y(0) = y_0, \quad y'(0) = y_1$$

represents the damped oscillation with viscoelastic intrinsic damping of oscillation [1, 2]. Several authors have studied the above equation and introduced numerical methods to approximate its solution (see, e.g [3, 4]). We propose a Nyström method for the global approximation of its solution. Such method is based on the discretization of the integral operator defining (2) by a suitable product quadrature rule obtained using Lagrange interpolation. Numerical tests showing the performances of the method will be presented, as well as comparisons with other numerical methods.

References

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15:30–15:50 An RBF-based Interface Capturing Method for Two-Phase Flows

Cecile Piret (Michigan Technological University, USA)

Abstract. Many applications in the natural and applied sciences involve the solution of partial differential equations (PDEs) on surfaces.Application areas for PDEs on static surfaces include image processing, biology, and computer graphics.
Applications for PDEs on moving surfaces also occur frequently. Notable examples arise in biology, material science, fluid dynamics, and computer graphics.
There are three main categories of methods for solving PDEs on arbitrary surfaces: the methods rely either on parametrization, on an embedding, or on triangulation. Embedding-type methods have the advantage that the surface-constrained differential operators are identical to their ambient-space analog. One of the most common embedding methods is the closest point method (CPM), [1]. The surface is enclosed inside a thick layer of nodes that belong to a dense three-dimensional grid. Each one of these nodes takes the function value of the one associated with their closest point to the surface, implicitly imposing that the normal derivatives at each node is null. Under that constraint, the surface Laplacian (for instance) is equivalent to its $\mathbf{R}^3$ analog.
Another embedding method called the Radial Basis Functions Orthogonal Gradients method (RBF-OGr) was introduced in [2]. This method is different as every computation is performed on the surface only, including the null normal derivative constraints. By using the RBF method, we take advantage of its meshfree character, which provides the flexibility to represent complex geometries in any spatial dimension while providing a high order of accuracy. Different versions of both the CPM and OGr methods, based on the RBF-based finite difference (RBF-FD) method, have been shown to be powerful tools to solve PDEs on static surfaces, [3,4,5].
This talk bears on the numerical solution of PDEs on *evolving* smooth curves and surfaces. We will introduce an RBF-OGr based method that captures the dynamics of a two-phase flow interface. A number of examples will be presented to illustrate the method.

References

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16:00–16:20 On the solution of elliptic PDEs through the multinode Shepard method

Filomena Di Tommaso (University of Calabria, ITALY)

Abstract. In this talk, we discuss the application of the multinode Shepard method [1] to numerically solve elliptic Partial Differential Equations (PDEs) equipped with various conditions at the boundary of domains of different shapes [2]-[4]. In particular, the multinode Shepard method is proposed to solve elliptic PDEs with Dirichlet and/or Neumann boundary conditions. In the case of discontinuous boundary conditions, it is necessary to design a method which is able to capture the singularities on the boundary.
In line with [5], enlargement of the functional space spanned by the multinode Shepard functions is proposed to overcome the difficulties due to the singularity on the boundary. Numerical results are presented to make evidence of the accuracy and efficiency of the method.

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17:00–17:20 A Novel Two-dimensional Fractional Wavelet Transform

Ahmed I. Zayed (DePaul University, USA)

Abstract. The subject of fractional integral transforms started in the early 1980's with the publication of Namias's paper on the fractional Fourier transform [1]. The fractional Fourier transform is a generalization of the Fourier transform that depends on an angle α and reduces to the standard Fourier transform for $\alpha = \pi/2$. It is given by

$$\mathcal{F}_\alpha[f](y) = \int_{\mathbb{R}} f(x)\mathcal{K}_\alpha(x,y)dx, \tag{1}$$

where $\mathcal{K}_\alpha(x,y)$ is given by

$$\mathcal{K}_\alpha(x,y) = \begin{cases} \sqrt{\frac{1-i\cot\alpha}{2\pi}} e^{-\frac{i}{2}(x^2+y^2)\cot\alpha + ixy\csc\alpha}, & \alpha \neq n\pi, \\ \frac{1}{\sqrt{2\pi}} e^{ixy}, & \alpha = \frac{\pi}{2}, \\ \delta(x-y), & \alpha = 2n\pi, \\ \delta(x+y), & \alpha = (2n-1)\pi, n \in \mathbb{Z}. \end{cases} \tag{2}$$

Namias's work, which has many applications in optics and imaging [2], was confined to one dimension, but it was later extended to higher dimensions using tensor products of one-dimensional transforms.

Nowadays there are fractional versions of many integral transforms, such as fractional Hankel, fractional Jacobi, fractional Mellin, fractional Radon, and fractional wavelet transforms [3]. Their extensions to higher dimensions were done in different ways.

Recall that a mother wavelet ψ is a function in $L^2(\mathbb{R})$ which satisfies the condition

$$0 \neq C_\psi = \int_{\mathbb{R}} \frac{|\hat{\psi}(w)|^2}{|w|} dw < \infty,$$

where $\hat{\psi}$ is the Fourier transform of ψ. The one-dimensional wavelet transform of $f \in L^2(\mathbb{R})$ with respect to the wavelet ψ is defined by

$$W_\psi f(x,s) = \frac{1}{\sqrt{s}} \int_{\mathbb{R}} f(t) \overline{\psi\left(\frac{t-x}{s}\right)} dt, \forall x \in \mathbb{R}, s \in \mathbb{R} \setminus \{0\},$$

where $\overline{\psi\left(\frac{t-x}{a}\right)}$ is the complex conjugate of $\psi\left(\frac{t-x}{a}\right)$.

In this talk we will discuss a new extension of the fractional wavelet transform to two dimensions that is not a tensor product of two one-dimensional transforms [4].

References

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[2] H. M. Ozakdas, M. A. Kutay, D. Mendlovic, *Introduction to the fractional Fourier transform and its applications, in Advances in Imaging Electronics and Physics*, New York: Academic, (1999).

[3] D. Mendlovic, Z. Zalevsky, D. Mas, J. Garcia, C. Ferreira, *Fractional wavelet transform*, Applied optics, 36 (20) (1997), 4801–4806.

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17:30–17:50 Novel wavelet systems for image processing applications

Mariantonia Cotronei (University Mediterranea of Reggio Calabria, ITALY)

Abstract. Wavelets have played a significant role in approximation theory, signal processing, and various other contexts for several decades.

The purpose of this talk is to illustrate the research which has recently been carried out by the author and collaborators in the direction of exploring new strategies for creating novel wavelet systems, particularly in the multivariate setting, with desirable properties in applications.

We present a tensor product argument that enables the explicit construction of filterbanks associated with a non-separable bivariate orthonormal wavelet system and arbitrary scaling matrices [1]. Such constructions are particularly useful in the implementation of a multiple multiresolution analysis [2]. In such a framework, which consists in a generalization of the discrete wavelet transform, the analysis and synthesis steps are implemented in terms of a composition of different (non-separable) filter operators and scaling matrices. This allows for a directional adaptation of the process to the image data and a proper detection of their singularities along curves.

Additionally, we present an extension of the polynomial wavelets proposed in [3] to the bivariate setting and the implementation of a fast algorithm for processing images. Preliminary results [4] demonstrate the potential of such novel bases in image compression problems.

References

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18:00–18:20 Accurate quasi interpolation via $m-$harmonic B-splines
Milvia Rossini (University of Milano - Bicocca, ITALY)

Abstract. The aim of this talk, is to provide a review on the present possible application of various kinds of polyharmonic B-splines, discuss different constructions and present a simple procedure that provides cardinal quasi-interpolation operators with high approximation order.

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