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Hyperbolic PDEs: Analytical Techniques and Applications

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Topic: The techniques of hyperbolic partial differential equations, in particular (of systems of) conservation laws, are the main topic of the special session "Hyperbolic PDEs: Analytical Techniques and Applications". The analytical results presented comprise also the case of mixed systems, in which conservation or balance laws are coupled also with ordinary differential equations, or equations of other types. The motivations for the tackled problems typically originate in very specific problems, suggested by a wide variety of applications such as fluid dynamics, epidemiology or biology and, mostly, the modeling of vehicular traffic flows.

Structure: This session is scheduled on July 23-24. It consists of 10 talks (each 45 minutes long followed by a 15-minute break).

Schedule and Abstracts

July 23, 2024

11:00–11:45 Going Forward and Backward in Time
in Conservation Laws and Hamilton-Jacobi Equations
Rinaldo M. Colombo (University of Brescia, ITALY)

Abstract. In the scalar one dimensional case, consider a conservation law and a Hamilton - Jacobi equation, i.e.,

$$\partial_t u + \partial_x f(x, u) = 0 \qquad \text{and} \qquad \partial_t U + f(x, \partial_x U) = 0\,.$$

The evident correspondence between the two equations is in deep contrast with the differences between their classical definitions of solutions. Indeed, Kružkov definition of entropy solution to a conservation law and Crandall - Lions definition of viscosity solution to a Hamilton - Jacobi equation appear as completely unrelated.

This talk presents first a framework where the well posedness of both equations can be proved and the correspondence between their solutions can be rigorously established. Then, given a time $T > 0$ and profiles $w \in \mathbf{L}^\infty(\mathbb{R}; \mathbb{R})$ or $W \in \mathbf{Lip}(\mathbb{R}; \mathbb{R})$, we characterize the two sets of initial data that, for the two equations, evolve into w or W at time T.

The explicit presence of the space variable in the flux or Hamiltonian f significantly enriches the theory, as shown by an explicit example. Applications to vehicular traffic management and data reconstruction are also considered.

References

[1] R.M. Colombo, V. Perrollaz, *Initial Data Identification in Conservation Laws and Hamilton - Jacobi Equations*, Journal de Mathématiques Pures et Appliquées, 9, 138, 1-27, 2020.

[2] R.M. Colombo, V. Perrollaz, *Localized Inverse Design in Conservation Laws and Hamilton - Jacobi Equations*, Preprint, 2024.

[3] R.M. Colombo, V. Perrollaz, A. Sylla, *Peculiarities of Space Dependent Conservation Laws: Inverse Design and Asymptotics*, to appear on the proceedings of the XVIII International Conference on Hyperbolic Problems: Theory, Numerics, Applications, 2023.

[4] R.M. Colombo, V. Perrollaz, A. Sylla, $\mathbf{L}^\infty$ *Stationary Solutions to Non Homogeneous Conservation Laws*, to appear on the proceedings of the XVIII International Conference on Hyperbolic Problems: Theory, Numerics, Applications, 2023.

[5] R.M. Colombo, V. Perrollaz, A. Sylla, *Initial Data Identification in Space Dependent Conservation Laws and Hamilton - Jacobi Equations*, to appear on Communications in Partial Differential Equations, 2024.

In collaboration with: Vincent Perrollaz (University of Tours, FRANCE) and Abraham Sylla (University of Brescia, ITALY).

12:00–12:45 A Nonlocal Version of Aw-Rascle-Zhang System
Debora Amadori (University of L'Aquila, ITALY)

Abstract. In this talk, we consider a macroscopic model for traffic flow. The model stems from the classical Aw-Rascle-Zhang system (ARZ) in which the pressure, describing the anticipation term along a trajectory, is replaced by a nonlocal version of it. The resulting system is an Euler-alignment pressureless system with a non-symmetric interaction kernel.

We tackle the study of weak solutions for this system through a sticky-particle approach, the difficulty being in the occurrence of non-conservative terms due to the singularity of the interaction kernel.

In collaboration with: Felisia A. Chiarello (University of L'Aquila, ITALY) and Gianmarco Cipollone (University of L'Aquila, ITALY).

14:30–15:15 Nonlocal Multi-D Systems of Hyperbolic Equations
Mauro Garavello (University of Milano-Bicocca, ITALY)

Abstract. We consider the following multi-D non linear system of hyperbolic equations

$$\begin{cases} \partial_t \rho_i + \nabla \cdot (\rho_i \, V_i(\nabla \rho * \eta)) = 0 & (t, x) \in \mathbb{R}_+ \times \mathbb{R}^n \\ \rho_i(0, x) = \rho_{o,i}(x) & x \in \mathbb{R}^n \end{cases} \qquad i \in \{1, \ldots, m\} \tag{1}$$

with non-local terms in the flux functions. Here $t \in \mathbb{R}_+$ is the time, $x \in \mathbb{R}^n$ is the space variable, $\rho = (\rho_1, \ldots, \rho_m)$ denotes the vector of time dependent densities of m populations defined on the whole space $\mathbb{R}^n$, $(\rho_{o,1}, \ldots, \rho_{o,m})$ are the initial conditions, and V_i, for $i \in \{1, \ldots, m\}$, represents the velocity function for the i-th population. We assume that each V_i depends on the $n \times m$ matrix $(\nabla \rho * \eta)$, where the ji entry is

$$(\nabla \rho(t) * \eta(x))_{ji} = \partial_{x_j} (\rho_i(t) * \eta)(x) = \partial_{x_j} \int_{\mathbb{R}^n} \rho_i(t, x - y) \eta(y) dy,$$

for $j \in \{1, \ldots, n\}$ and $i \in \{1, \ldots, m\}$ and $\eta : \mathbb{R}^n \to \mathbb{R}$ is a smooth kernel.

The model (1) is of macroscopic type and is able to describe different behaviors typically emerging in population dynamics. Indeed different shapes of the kernel function η and of the velocity vectors V_i may result for example in an aggregation phenomenon, with the possible formation of clusters (or opinions), or in a segregation of the various populations. An important role is played by the support of the kernel function, which correspond to the visual range of the individuals. In the talk we present several numerical integrations showing various behaviors of (1), together with its well posedness and some analytic qualitative properties.

In collaboration with: Rinaldo M. Colombo (University of Brescia, ITALY).

15:30–16:15 Stabilization of Evolution Systems
Amaury Hayat (Ecole des Ponts Paristech, FRANCE)

Abstract. In this talk, we will discuss recent progresses in stabilization of PDE systems. We will begin by introducing a recent approach –known as Fredholm backstepping (or F-equivalence)– which has significantly improved in the last two years and has demonstrated a surprisingly remarkable efficacy [1, 2, 3]. Instead of attempting directly to solve the problem, the principle of the F-equivalence is to allow the PDE system to be invertibly transformed into a simpler PDE system whose stability is already established.

Next, we will discuss practical applications: stabilizing hyperbolic PDEs to control navigable rivers and traffic flows. They are particular examples of what is called *density-velocity* systems, which consist in two equations: a conservation of mass and a balance of momentum or energy [5].

We will present some results obtained in the last few years and show how abstract mathematical concepts, like entropic solutions, can have tangible impacts in real-world scenarios [4].

Finally, if time allows, we will dedicate a few minutes to explore how AI models can be trained to learn a mathematical intuition, in particular in stabilization of evolution systems.

References

[1] Jean-Michel Coron, Amaury Hayat, Shengquan Xiang, and Christophe Zhang. Stabilization of the linearized water tank system. *Archive for Rational Mechanics and Analysis*, 244(3):1019–1097, 2022.

[2] Ludovick Gagnon, Amaury Hayat, Shengquan Xiang, and Christophe Zhang. Fredholm backstepping for critical operators and application to rapid stabilization for the linearized water waves. *arXiv preprint arXiv:2202.08321*, 2022.

[3] Ludovick Gagnon, Amaury Hayat, Shengquan Xiang, and Christophe Zhang. Fredholm transformation on laplacian and rapid stabilization for the heat equation. *Journal of Functional Analysis*, 283(12):109664, 2022.

[4] Amaury Hayat, Thibault Liard, Francesca Marcellini, and Benedetto Piccoli. A multiscale second order model for the interaction between av and traffic flows: analysis and existence of solutions. 2021.

[5] Amaury Hayat and Peipei Shang. Exponential stability of density-velocity systems with boundary conditions and source term for the H^2 norm. *Journal de mathématiques pures et appliquées*, 153:187–212, 2021.

17:00–17:45 Smoothing Effect and Particle Approximation for a Nonlocal Conservation Law
Marco Di Francesco (University of L'Aquila, ITALY)

Abstract. Conservation laws with nonlocality in the flux appear in many applied contexts such as traffic flow, sedimentation processes, supply chain modelling, and crowd movements. In this talk we consider the following version of a one-dimensional, nonlocal scalar conservation law (extensively studied in the literature recently, see for example the papers below)

$$\partial_t \rho + \partial_x(\rho W[\rho]) = 0 \tag{2}$$

where $W : \mathcal{P}(\mathbb{R}) \to L^\infty(\mathbb{R})$ is a nonlocal operator defined by

$$W[\rho](x) = v(V * \rho)(x)$$

with $\mathcal{P}(\mathbb{R})$ being the space of probability measures on $\mathbb{R}$. We assume that $V \in L^1 \cap L^\infty(\mathbb{R}; [0, +\infty))$ satisfies $\mathrm{supp}(V) \subset (-\infty, 0]$, $V \in \mathrm{Lip}((-\infty, 0])$, V left continuous at zero with $V(0^-) > 0$. The velocity map $v : [0, +\infty) \to \mathbb{R}$ is supposed to be Lipschitz continuous with $v'(\rho) \leq -b$ (almost everywhere) for some $b > 0$ for all $\rho \geq 0$.

We prove that the Cauchy problem for (2) is well posed for initial data $\rho_0 \in \mathcal{P}(\mathbb{R})$, more precisely in the 2-Wasserstein space. Moreover, we prove that all solutions satisfy an instantaneous smoothing effect from $\mathcal{P}(\mathbb{R})$ to $L^\infty(\mathbb{R})$. Finally, we provide a deterministic particle approximation of (2) in terms of solutions to a follow-the-leader type ODE system.

References

[1] A. Bressan, W. Shen *On traffic flow with nonlocal flux: A relaxation representation.*, Archive for Rational Mechanics and Analysis, 2020.

[2] M. Colombo, G. Crippa, E. Marconi, and L.V. Spinolo, *Nonlocal traffic models with general kernels: Singular limit, entropy admissibility, and convergence rate.*, Archive for Rational Mechanics and Analysis, 2023.

[3] A. Keimer and L. Pflug. *On the singular limit problem for a discontinuous nonlocal conservation law.* Nonlinear Analysis, 2023.

In collaboration with: Simone Fagioli (University of L'Aquila, ITALY) and Emanuela Radici (University of L'Aquila, ITALY).

July 24, 2024

11:30–12:15 A 2×2 System of Conservation Laws with Discontinuous Flux and Traffic Applications
Felisia A. Chiarello (University of L'Aquila, ITALY)

Abstract. We consider a Follow-the-Leader-type microscopic system and prove the rigorous micro-macro convergence in the many particle limit in presence of vacuum to weak solutions of a second order system that includes the hydrodynamic traffic flow model introduced in [3] with space discontinuous flux, that here we call CGST model, and the classical ARZ model in [1]-[4] . The general system reads

$$\begin{cases} \partial_t \rho + \partial_x \left(c\, V(h)\, \rho \right) = 0, \\ \partial_t \left(\rho \left(h + p(\rho) \right) \right) + \partial_x \left(c\, V(h)\, \rho \left(h + p(\rho) \right) \right) = 0. \end{cases}$$

It finds its relevance in the modeling of vehicular traffic. In this context, $t \geq 0$ denotes the time, $x \in \Re$ the space, $\rho = \rho(t, x) \geq 0$ the density, $p = p(\rho) \geq 0$ is the pressure function, $c = c(x) > 0$ is a discontinuous function that mimics the road capacity, and $V \geq 0$ represents the speed law. The interpretation of $h = h(t, x) \geq 0$ depends on the specific model under consideration. For instance, the function h represents the mean headway in the model proposed in [3] and the velocity in the ARZ model, see [1]-[4]. We complement our result with numerical simulations of the particle method compared with some macroscopic approximate solutions obtained with the Lax-Friedrichs scheme.

References

[1] A. Aw and M. Rascle. *Resurrection of "second order" models of traffic flow.* SIAM Journal on Applied Mathematics, 60(3):916–938, 2000.

[2] F. A. Chiarello, S. Fagioli, M. Rosini. *A 2 x 2 system of conservation laws with discontinuous flux and traffic applications.* In preparation.

[3] F. A. Chiarello, S. Göttlich, T. Schillinger, A. Tosin *Hydrodynamic traffic flow models including random accidents: A kinetic derivation.* Volume 22 (2024), number 3, pages: 845 - 870. Communications in Mathematical Sciences.

[4] H. Zhang. *A non-equilibrium traffic model devoid of gas-like behavior.* Transportation Res. Part B, 36(3):275–290, 2002.

In collaboration with: Simone Fagioli (University of L'Aquila, ITALY) and Massimiliano Rosini (University of Chieti-Pescara, ITALY).

12:15–13:00 Unique Solutions to Hyperbolic Conservation Laws with a Strictly Convex Entropy
Graziano Guerra (University of Milano-Bicocca, ITALY)

Abstract. Consider a strictly hyperbolic $n \times n$ system of conservation laws, where each characteristic field is either genuinely nonlinear or linearly degenerate. In this standard setting, it is well known that there exists a Lipschitz semigroup of weak solutions, defined on a domain of functions with small total variation. If the system admits a strictly convex entropy, we give a short proof that every entropy weak solution taking values within the domain of the semigroup coincides with a semigroup trajectory [2]. The result shows that the assumptions of "Tame Variation" or "Tame Oscillation" (see [1] and references therein), previously used to achieve uniqueness, can be removed in the presence of a strictly convex entropy.

Combined with a compactness argument, the result yields a uniform convergence rate, $\rho(\varepsilon) \to 0$ as $\varepsilon \to 0$, for a very wide class of approximation algorithms. Some partial estimates on $\rho(\varepsilon)$ are given.

References

[1] A. Bressan and P. Goatin, *Oleinik type estimates and uniqueness for $n \times n$ conservation laws*, J. Differential Equations, 156 (1999), 26–49.

[2] A. Bressan and G. Guerra, *Unique solutions to hyperbolic conservation laws with a strictly convex entropy*, J. Differential Equations, 387 (2024), 432–447.

In collaboration with: Alberto Bressan (Penn State University, USA).

14:30–15:15 A Strongly Coupled PDE-ODE Model in the Presence of Discontinuity in the Flux
Hossein Nick Zinat Matin (University of California, Berkeley, USA)

Abstract. In this work, we introduce a system of PDE-ODEs with discontinuity in the flux function with application to traffic flow:

$$\begin{cases} \partial_t \rho + \partial_x [f(\gamma, \rho)] = 0 & , x \in \mathbb{R},\ t \in \mathbb{R}_+ \\ \rho(0,x) = \rho_\circ(x) \\ f(\gamma,\rho) - \dot{y}\rho \le F(y) \\ \dot{y} = w(y,\rho) \end{cases}.$$

where ρ denotes the traffic density, f is the flux function and γ captures a variable speed limit. Strongly coupled PDE-ODE, models have been exploited in various works to present the effect of controllers in regulating the behavior of traffic flow, among other applications. In such settings, the mass behavior of traffic is modeled through PDEs and the dynamics of controllers are modeled by a system of ODEs. The interaction of ODEs with other fronts can create a so called non-classical shock that introduces some complexities in proving the well-posedness of the problem. The discontinuity in the flux of conservation laws can arise from application point of view; e.g., changes in maximum speed in traffic flow applications. Such discontinuities cause an oscillatory behavior in the system and consequently lack of uniform bounded variation of the solutions which will restrict using the conventional tools to establish the compactness theorem.

We use the wavefront tracking method to show the existence of solution. In doing so, first we define a Riemann solution that explains both the non-classical shock and the jump in the flux. As the system lacks bounded variation, we prove the convergence of the approximate solutions using a homeomorphism. We will investigate different cases that can arise as the result of interaction of the controllers' trajectory with other waves under this setting and will discuss the potential complexities. Eventually, using the properties of the solutions in the homeomorphic space, we will show the compactness theorem and subsequently the existence of the solution.

In collaboration with: Maria Laura Delle Monache (University of California, Berkeley, USA).

15:30–16:15 Scalar Approach to ARZ-Type Systems of Conservation Laws
Abraham Sylla (University of Brescia, ITALY)

Abstract. We are interested in 2×2 systems of conservation laws of special structure, including generalized Aw-Rascle and Zhang (GARZ) models for road traffic. The simplest representative is the Keytz-Kranzer system, where one equation is nonlinear and not coupled to the other, and the second equation is a linear transport which coefficients depend on the solution of the first equation.

In GARZ systems, the coupling is stronger, they do not have the "triangular" structure of Keytz-Kranzer. The systems we consider take the form

$$\begin{cases} \partial_t \rho + \partial_x \left(f(w,\rho)\right) = 0 \\ \partial_t \left(w\rho\right) + \partial_x \left(w f(w,\rho)\right) = 0. \end{cases}$$

We claim that it makes sense to address these systems *via* a kind of splitting approach. Indeed, in 2008, Panov proposed a robust framework for solving linear transport equations with divergence free coefficients. Our idea is to use this framework for the second equation of GARZ systems, and to exploit the scalar discontinuous flux framework for the first equation of the system.

References

[1] B. Andreianov, A. Sylla, *Existence analysis and numerical approximation for a second order model of traffic with orderliness marker*, M3AS (2022), 1295–1348.

[2] B.L. Keyfitz, H.C. Kranzer, *A system of nonstrictly hyperbolic conservation laws*, Arch. Ration. Mech. Anal., 72(1979/80), 219—241.

[3] E.Y. Panov, *Generalized solutions of the Cauchy problem for a transport equation with discontinuous coefficients*, Springer New York, New York, NY (2008), 23–84.

In collaboration with: Boris Andreianov (University of Tours, FRANCE).

17:00–17:45 Hybrid and Multiscale Models for Vehicular Traffic
Benedetto Piccoli (Rutgers University, USA)

Abstract. Classical models for vehicular traffic include the celebrated Lighthill-Whitham-Richards one:

$$\rho_t + f(\rho)_x = 0 \tag{3}$$

where $\rho \in [0, \rho_{max}]$ is the car density, the flux verifies $f(\rho) = \rho\, v(\rho)$, and $v(\rho)$ is the average velocity. These models proved to be successful in traffic modeling and monitoring. New technologies advances allow the development of new control mechanism, in particular via autonomous vehicles. To model this Delle Monache and Goatin proposed the following PDE=ODE coupled system:

$$\begin{cases} \rho_t + f(\rho)_x = 0 \\ \dot{y}(t) = \min\{\omega(t), v(\rho(t, y(t)+))\} \\ f(\rho(t, y(t))) - \dot{y}(t)\rho(t, y(t)) \leq F_\alpha(\dot{y}(t)) \end{cases} \tag{4}$$

where $f(\rho) = \rho\, v(\rho)$, the control $\omega \in [0, U]$ indicates a speed chosen by the bottleneck, say an autonomous vehicle or a vehicle with autonomous cruise control, $v(\rho(t, y(t)+)) = \lim_{h\to 0, h\geq 0} v(\rho(t, y(t)+h))$, and F_α is given by:

$$F_\alpha(\dot{y}(t)) := \max_\rho (\alpha f(\rho/\alpha) - \rho\dot{y}(t)), \qquad \alpha \in\,]0, 1[. \tag{5}$$

In this talk we report various results on the well-posedness of such system, and its extension to include a two equation PDE traffic model and a multilane setting.

References

[1] M.L. Delle Monache, P. Goatin, *Scalar conservation laws with moving constraints arising in traffic flow modeling: an existence result.*, Journal of Differential equations 257.11 (2014): 4015-4029.

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