AMS-UMI International
Joint Meeting 2024
Palermo, July 23–26, 2024

Advances in Geometric Control Theory and Subelliptic PDEs
Special Session A16

Nicolò Forcillo
Michigan State University, USA

Cristian Mendico
University of Roma Tor Vergata, ITALY

In the last decade, attention to sub-Riemannian geometries has rapidly spread in several directions. These are concerned with the geometrical aspects of control problems and the analytical properties of PDEs defined on such anisotropic structures. These problems arise from different natural models, such as the geometric theory of several complex variables, curvature problems, diffusion processes, human vision, and multi-agent dynamics. All of these models have in common the fact that their ellipticity directions span subspaces of dimension strictly less than the dimension of the state space, and all the remaining directions are recovered from commutators. This implies that the underlying geometric structure of the state space is of an anisotropic type, which plays a crucial role in the controllability properties of the dynamical system, as well as in the analysis and regularity of solutions to PDEs.

This special session aims to gather researchers interested in the study of geometric optimal control theory, linear/nonlinear elliptic and parabolic PDEs, and their connection represented by the sub-Riemannian setting. During the conference, we will primarily focus on the following themes: Hamilton-Jacobi equations and viscosity solutions tailored to the new geometric framework, qualitative and quantitative aspects of solutions to subelliptic PDEs such as Liouville properties, Harnack-type inequalities, and maximum principles driven by possibly degenerate elliptic and parabolic, linear and nonlinear, subelliptic operators.

We plan to divide the talks in this session into plenary communications, held by senior researchers aiming to present not only known results, but also open problems to inspire discussions and collaborations, and general communications aiming at spreading the most recent results.

SCHEDULE AND ABSTRACTS

July 23, 2024

11:00–11:45 Giovanna Citti (Università di Bologna, ITALY)

Title. TBA
Abstract.

12:00–12:20 Marco Caponigro (Università di Roma Tor Vergata, ITALY)

Title. A geometric control method for bilinear quantum systems

Abstract. We consider a bilinear control systems of the type

$$\dot{x}(t) = (A + u(t)B)x(t)$$

where the state x belongs to some complex infinite dimensional Hilbert space X, the (possibly unbounded) linear operators A and B are skew-adjoint and the control u is a real valued function. Such systems arise, for instance, in quantum control with the bilinear Schrödinger equation. Our goal is to describe the *reachable set*, the set of points attainable in finite time from an initial datum using controls in a certain class. We say that the system is *controllable* if the reachable set is the whole space X.

In [1] the authors showed that if the operator B is bounded the rechable set with L^p controls, $p > 1$, is contained in a countable union of compact subsets of X. Hence, when X is infinite dimensional, this result represents an obstruction to exact controllability. We present a sufficient condition on the system for approximate controllability with piecewise constant controls [3]. On the other hand we extend, under suitable hypotheses on the commutator of the operators A and B, the definition of solution for controls in the set of Radon measures, obtaining precise *a priori* energy estimates on the solution [2] and, as a consequence, upper estimates on the reachable sets. As a particular case we have an extension to L^1 of the noncontrollability result of [1].

References

[1] Ball, J. M., Marsden J. E., & Slemrod, M. "Controllability for distributed bilinear systems." *SIAM Journal on Control and Optimization* 20.4 (1982): 575-597.

[2] Boussaïd, N., Caponigro, M. & Chambrion, T. "Regular propagators of bilinear quantum systems." *Journal of Functional Analysis* 278.6 (2020): 108412.

[3] Caponigro, M., & Sigalotti, M. "Exact controllability in projections of the bilinear Schrödinger equation." *SIAM Journal on Control and Optimization* 56.4 (2018): 2901-2920.

12:30–12:50 Annalaura Rebucci (Max Planck Institute for Mathematics, GERMANY)

Title. On the Harnack inequality for degenerate Kolmogorov operators with non-smooth coefficients

Abstract. After recalling the classical mean value formulas for uniformly parabolic operators with smooth coefficients, we deal with strongly degenerate Kolmgorov-type operators with less regular coefficients. More precisely, following [2], we combine the Mean Value formulas established in [1] with a "descent method" due to Kuptsov to obtain formulas with improved kernels. We then exploit the mean value formulas to prove an invariant Harnack inequality and a strong maximum principle for classical solutions to a large class of subelliptic PDEs. Our analysis is carried out in the non-Euclidean framework of the associated Lie group. We point out that the proofs of our main results only rely on the classical theory developed for harmonic functions.

References

[1] D. Pallara, S. Polidoro, *Mean value formulas for classical solutions to subelliptic evolution equations in stratified Lie groups*, preprint arXiv (2023).

[2] A. Rebucci, *Harnack inequality and maximum principle for degenerate Kolmogorov operators in divergence form*, J. Math. Anal. Appl., 538(2) (2024).

14:30–15:15 Davide Barilari (Università degli Studi di Padova, ITALY)

Title. Strichartz estimates in the Heisenberg group

Abstract. In this talk I will discuss Strichartz estimates on the Heisenberg group for the linear Schrödinger and wave equations involving the sublaplacian. The Schrödinger equation on the Heisenberg group is an example of a totally non-dispersive evolution equation: for this reason the classical approach that permits to obtain Strichartz estimates from dispersive estimates is not available.

Our approach, inspired by the Fourier transform restriction method initiated by Tomas and Stein, is based on Fourier restriction theorems, using the non-commutative Fourier transform on the Heisenberg group. It enables us to obtain also an anisotropic Strichartz estimate for the wave equation, for a larger range of indices than was previously known.

If time permits, I will present results on the Engel group and more general Carnot groups.

References

[1] H. Bahouri, D. Barilari, I. Gallagher, *Strichartz estimates and Fourier restriction theorems on the Heisenberg group* , Journal of Fourier Analysis and Applications, 2021.

[2] H. Bahouri, D. Barilari, M. Léautaud, I. Gallagher, *Spectral summability for the quartic oscillator with applications to the Engel group* , Journal of Spectral Theory, 2023.

15:30–15:50 Paola Mannucci (Università degli Studi di Padova, ITALY)

Title. Second order mean field games on homogeneous Lie groups

Abstract. The model of the mean field games (MFG) describes interactions among a very large number of identical agents. Evolutive second order MFGs occur when the time horizon is finite and the dynamics of the agents is stochastic. They are modelized by a system of two coupled parabolic equations: a backward in time Hamilton-Jacobi equation and a forward in time Fokker-Plank equation describing respectively the optimal cost of a generic agent and the distribution of the whole population.

We study existence of classical solutions to mean field games systems defined on homogenous Lie group. We consider an homogeneous Lie group, endowed with a family of dilations, which can be identified with $\mathbb{R}^d$. Let $\{X_1, \dots, X_m\}$ (with $m < d$) be a family of vector fields which satisfies the Hörmander condition; in particular, together with their commutators, these vector fields generate the Lie group. We consider second order mean field games systems where the differential operators are given in terms of these vector fields and where the couplings are strongly regularizing. In our model, each agent can move only along the directions generated by $X_1, \dots, X_m$ but it can still reach every point due to the Hörmander condition.

In order to obtain the existence of solutions to these mean field games, we first study existence and uniqueness of the subelliptic Fokker-Planck equation and separately of the subelliptic Hamilton-Jacobi equation.

References

[1] P. Mannucci, C. Marchi and C. Mendico, *Semi-linear parabolic equations on homogenous Lie groups arising from mean field games*, Math. Ann. (2024).

16:00–16:20 Valentina Franceschi (Università degli Studi di Padova, ITALY)

Title. Mean value formulas for surfaces in Grushin spaces

Abstract. In this talk, we consider n-dimensional Grushin spaces, where a Riemannian metric degenerates along a line in the space, resulting in a sub-Riemannian structure. We discuss the validity of (sub-)mean value property for (sub-)harmonic functions on hypersurfaces within Grushin spaces of dimension $n > 2$. Our interest is driven by the classical counterpart: mean value formulas for harmonic functions on surfaces in the Euclidean setting are crucial for establishing the Bombieri-De Giorgi-Miranda gradient bound, which, in turn, plays a central role in the classical regularity theory. We conclude by presenting remarks and open questions about the

regularity theory of minimal surfaces within this sub-Riemannian framework, which is yet to be established.

References

[1] V. Franceschi, R. Monti, and A. Socionovo, *Mean value formulas on surfaces in Grushin spaces*, Annales Fennici Mathematici, 49(1) (2024), 241–255

16:00–16:20 Daniela Di Donato (Università di Pavia, ITALY)

Title.Rectifiability in Carnot groups

Abstract. Intrinsic regular surfaces in Carnot groups play the same role as C^1 surfaces in Euclidean spaces. As in Euclidean spaces, intrinsic regular surfaces can be locally defined in different ways: e.g. as non critical level sets or as continuously intrinsic differentiable graphs. The equivalence of these natural definitions is the problem that we are studying. Precisely our aim is to generalize some results proved by Ambrosio, Serra Cassano, Vittone [1] valid in Heisenberg groups to the more general setting of Carnot groups.

References

[1] L. Ambrosio, F. Serra Cassano, D. Vittone *Intrinsic regular hypersurfaces in Heisenberg groups*, J. Geom. Anal. 16, 2006, 187?232.

16:00–16:20 Giacomo Giovannardi (Università di Firenze, ITALY)

Title. The asymptotic p-Poisson equation as $p \to \infty$ in Carnot-Carathéodory spaces

Abstract. In this talk we will deal with the asymptotic behavior of solutions to the subelliptic p-Poisson equation as $p \to +\infty$ in Carnot Carathéodory spaces. In particular, introducing a suitable notion of differentiability, we extend the celebrated result of Bhattacharya, DiBenedetto and Manfredi [1] and we prove that limits of such solutions solve in the sense of viscosity a hybrid first and second order PDE involving the $\infty-$Laplacian and the Eikonal equation.

References

[1] T. Bhattacharya, E. DiBenedetto and J. Manfredi, *Limits as $p \to \infty$ of $\Delta_p u_p = f$ and related extremal problems*, Rend. Sem. Mat. Univ. Politec. Torino, Fascicolo Speciale (1989), Nonlinear PDE's, 15-68.

July 24, 2024

11:30–12:15 Irina Markina (University of Bergen, GERMANY)

Title. Local invariants and geometry of the sub-Laplacian on H-type foliations

Abstract. Let (M, g) be a smooth, oriented, connected Riemannian manifold equipped with a Riemannian foliation with bundle-like complete metric g and totally geodesic leaves satisfying some additional symmetry conditions. The manifold is studied in the framework of sub-Riemannian geometry with bracket generating distribution transversal to the totally geodesic fibers. Equipping M with the Bott connection we find local invariants by studying the small-time asymptotics of the sub-Riemannanian heat kernel. We obtain the first three terms in the asymptotic expansion of the Popp volume for the pull-back of small sub-Riemannian balls. We address also the question of local isometry of M as a sub-Riemannian manifold and its tangent group.

This is the joint work with W. Bauer, A. Laaroussi (Leibnitz University of Hannover, Germany), S. Vega-Molino (University of Bergen, Norway)

References

[1] W. Bauer, I. Markina, A. Laaroussi, G. Vega-Molino, *Local Invariants and Geometry of the sub-Laplacian on H-type Foliations.* arXiv preprint 2209.02168

[2] F. Baudoin, E. Grong, L. Rizzi, G. Vega-Molino, *H-type Foliations.* Differential Geom. Appl. 85 (2022), Paper No. 101952, 25 pp.

12:30–12:50 Roberta Ghezzi (Università di Roma Tor Vergata, ITALY)

Title. Regularity theory and geometry of unbalanced optimal transport

Abstract. Using the dual formulation only, we show that the regularity of unbalanced optimal transport also called entropy-transport inherits from the regularity of standard optimal transport. We provide detailed examples of Riemannian manifolds and costs for which unbalanced optimal transport is regular. Among all entropy-transport formulations, Wasserstein-Fisher-Rao (WFR) metric, also called Hellinger-Kantorovich, stands out since it admits a dynamic formulation, which extends the Benamou-Brenier formulation of optimal transport. After demonstrating the equivalence between dynamic and static formulations on a closed Riemannian manifold, we prove a polar factorization theorem, similar to the one due to Brenier and Mc-Cann. As a byproduct, we formulate the Monge-Ampere equation associated with WFR metric, which also holds for more general costs. Last, we study the link between c-convex functions for the cost induced by the WFR metric and the cost on the cone. The main result is that the weak Ma-Trudinger-Wang condition on the cone implies the same condition on the manifold for the cost induced by WFR.

References

[1] T. Gallouët, R. Ghezzi and F.-X. Vialard, *Regularity theory and geometry of unbalanced optimal transport*, preprint hal-03498098v2

14:30–15:15 Fausto Ferrari (Università di Bologna, ITALY)

Title. Few counterexamples to ACF-Monotonicity formulas in Carnot groups

Abstract. In this talk, we discuss a way to obtain some counterexamples to the increasing behavior of the Alt-Caffarelli-Friedman monotonicity formula, see [1,2], in Carnot groups of step 2, see [4]. In addition, we present an overview of some statements concerning the different behavior of intrinsic harmonic functions in Carnot groups of step 2, like in the Heisenberg group $\mathbb{H}^1$, with respect to the well known monotonicity increasing behavior of the Alt-Caffarelli-Friedman functional of harmonic functions in the Euclidean setting. In particular, we introduce some results, contained in the papers, [3,4,5], obtained in collaboration with Nicolò Forcillo (Michigan State University), and then continued in [6], with Davide Giovagnoli (University of Bologna).

References

[1] H. W. Alt, L. Caffarelli, A. Friedman *Variational problems with two phases and their free boundaries*, Trans. Amer. Math. Soc. 282 (1984), no. 2, 431–461.

[2] L. Caffarelli, S. Salsa, *A geometric approach to free boundary problems*, Grad. Stud. Math. 68, American Mathematical Society, Providence, RI, 2005.

[3] F. Ferrari, N. Forcillo, *A new glance to the Alt?Caffarelli?Friedman monotonicity formula*, Math. Eng. 2 (2020), no. 4, 657–679.

[4] F. Ferrari, N. Forcillo, *A counterexample to the monotone increasing behavior of an Alt-Caffarelli-Friedman formula in the Heisenberg group*, Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl. 34 (2023), no. 2, 295–306 and https://doi.org/10.48550/arXiv.2203.06232

[5] F. Ferrari, N. Forcillo, *Alt-Caffarelli-Friedman monotonicity formula and mean value properties in Carnot groups with applications*, Boll Unione Mat Ital (2023). https://doi.org/10.1007/s40574-023-00393-5

[6] F. Ferrari, D. Giovagnoli, *Some counterexamples to Alt-Caffarelli-Friedman monotonicity formulas in Carnot groups*, https://doi.org/10.48550/arXiv.2401.07679

15:30–15:50 Francesca Anceschi (Università Politecnica Delle Marche, ITALY)

Title. Well-posedness results for Kolmogorov equations with applications to mean-field control problems for multi-agent systems

Abstract. In this talk, based on a joint project with G. Ascione, D. Castorina and F. Solombrino, we discuss some well-posedness results for Kolmogorov-Fokker-Planck equations with measurable coefficients in time and locally Hölder continuous coefficients in space with possibly unbounded drift terms and their application to the study of particle systems of the second order whose prototypical agent is driven by a McKean-Vlasov SDE, or by a Vlasov-Fokker-Planck PDE.

16:00–16:20 Eugenio Pozzoli (Universityé de Rennes 1, FRANCE)

Title. Small-time controllability of bilinear PDEs with infinite-dimensional Lie bracket methods

Abstract. This talk is devoted to some recent results of global approximate controllability, in arbitrarily small times, of bilinear Schrödinger and wave equations posed on boundaryless manifolds such as tori or euclidean spaces. The analysis of the controllability properties of these PDEs is very rich, and many results of large-time controllability exist in the literature. The focus of the presentation is thus on the possibility of controlling such PDEs in arbitrarily small times. Moreover, the main ideas behind the proofs of the results will also be presented: they can be interpreted as extensions, to this infinite-dimensional setting, of classical Lie bracket methods widely used in the literature to analyse finite-dimensional nonlinear control systems.

This talk is based on the results contained in the article [1] and in a forthcoming preprint in collaboration with Karine Beauchard (ENS Rennes).

References

[1] E. Pozzoli, *Small-time global approximate controllability of bilinear wave equations*, Journal of Differential Equations, 388 (2024), 421–438.

17:00–17:20 Fabio Camilli (SBAI, Università La Sapienza, ITALY)

Title. A Mean-Field Game network model for urban planning.

Abstract. We delve into a mathematical framework aimed at capturing the dynamic evolution of urban landscapes, shaped by the interplay between two entities: the workforce and business establishments. Visualizing the city as a network, where edges symbolize both residential zones and connectivity pathways, we explore how these populations vie for spatial occupancy while engaging in labor market transactions. Our formulated model is composed of a two-population Mean-Field Game system coupled with an Optimal Transport problem, showcasing the coalescence of complex dynamics. We establish the existence and uniqueness of solutions and supplement our findings with diverse numerical simulations.

17:00–17:20 Daria Ghilli (Università di Pavia, ITALY)

Title. Mean field control and Mean field games for multiple species models to maintain biodiversity.

Abstract. We propose a multi-agent model, in which biodiversity is valued both through the provisioning services it enables, and through its non-use value. Thus, the interactions between local decisions and local and global consequences are intimately linked. We are interested in the roles played by global interactions on individual decisions. More specifically, we consider a multi-agent model, where individual preferences depend on the relative utility of consumption and on biodiversity. In order to use the abundance-based characterization of biodiversity, species growth is defined through a Gompertz dynamic, which allows the inclusion of a density-dependent mortality relationship. We are interested in the behaviour of the system for a large number of agents and to this end we study the problem by mean field techniques. We compare the impacts of these decisions in two different scenarios: the first, the agents are competitive and hopefully reach a Nash equilibrium; the second, they cooperate, leave the choice of the control to a social planner and reach a social optimum. The goal of the model is to take into account for the relationship

between local damage to biodiversity, the local function of protection against environmental risks and the global role of prevention against global environmental disruption.

17:00–17:20 Chiara Cicolani (Università de L'Aquila, ITALY)

Title. Opinion dynamics of two populations with time-delayed coupling.

Abstract. We study a Hegselmann-Krause type opinion formation model for a system of two populations. The two groups interact with each other via subsets of individuals, namely the leaders, and natural time delay effects are considered. By using careful estimates of the system's trajectories, we are able to prove an asymptotic convergence to consensus result. Some numerical tests illustrate the theoretical result and point out some possible applications.

This kind of model can have applications in social sciences, economics, politics, and ecology. Indeed, it is reasonable to try reaching a global consensus among individuals of different countries, or different groups of individuals in the same country, about important questions such as, e.g., ecological behaviors, climate change's reasons, appropriate strategies to reduce CO_2 emissions, etc. The proof of a consensus result for models like that can be considered as a first insight for more quantitative studies aiming to design appropriate control strategies.

References

[1] E. Continelli and C. Pignotti, Consensus for Hegselmann-Krause type models with time variable time delays, *Math. Methods Appl. Sci.*, 46, (2023), 18916–18934.

[2] J. Haskovec, Direct proof of unconditional asymptotic consensus in the Hegselmann-Krause model with transmission-type delay, *Bull. London Math. Soc.*, 53, (2021), 1312–1323.

[3] A. Paolucci and C. Pignotti, Consensus strategies for a Hegselmann-Krause model with leadership and time variable time delay, J. Dynam. Differential Equations, published online (2023), https://doi.org/10.1007/s10884-023-10276-0.

[4] R. Solé and J. Bascompte, *Self-Organization in Complex Eco-systems*, Princeton University Press, Princeton, NJ, 2006.

E–mail: mendico@axp.mat.uniroma2.it.