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Analysis, control and inverse problems in climate sciences
Special Session B25

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The ongoing climate emergence and its consequences have emphasized the crucial role of a deep mathematical research on such topic. The fight against climate change, the study and possibly prediction of extreme events, the analysis of potential effects on the enviroment have nowadays attracted significant attention of scientists from a wide range of disciplines. In particular, the investigation of differential equations which describe such phenomena is certainly of fundamental importance: from the definition and analysis of reliable climate models, the study of the dependence on significant parameters, and finally the employment of such models to infer social and enviromental impacts of climate change.

The scope of this special session is to gather experts on the aforementioned subjects in order to present and discuss new mathematical developments on such topics. Our aim is to give an interdisciplinary overview of the problems connected to climate change and the related techniques arising in mathematical and numerical analysis, stochastic calculus, dynamical systems, ODEs and PDEs analysis.

Schedule and Abstracts

July 25, 2024

11:30–11:50 Nonautonomous Dynamical Systems for Climate Change & Climate Variability: An Application to a Simple Ocean Model

Michael Ghil (Ecole Normale Supérieure, FR)

Abstract The theory of nonautonomous and random dynamical systems (NDSs and RDSs) provides an appropriate mathematical setting for the study of time-dependent forcing, both natural and anthropogenic, upon a climate system characterized by intrinsic variability [1]. In this theory, the forward attractors of autonomous dynamical systems are replaced by pullback and random attractors (PBAs and RAs) and classical bifurcations by "tipping points." Over the last decadeand-a-half, these relatively novel concepts have been applied to a number of simple climate models, atmospheric, oceanic and coupled [2]. Important insights into the study of PBAs and RAs arising from climate dynamics have been provided by novel tools from algebraic topology [3,4]. These tools have led to the introduction and analysis of topological tipping points and we present them here as applied to a simple model of the wind-driven double-gyre ocean circulation [5]. The model is a low-order approximation of a spectral quasigeostrophic model for the subtropical and subpolar gyres of the North Atlantic or North Pacific ocean basin, subject to time varying zonal winds [6]. The recent tools from algebraic topology applied to it are Branched Manifold Analysis through Homologies (BraMAH) and the Templex, which combines the complex underlying BraMAH with a directed graph that captures the flow in the dynamical system's phase space [4].

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12:00–12:20 Typicality of extreme events in the climate system – A large deviation theory perspective –

Vera Melinda Galfi (Vrije Universiteit, NL)

Abstract In this presentation I will shortly present the main tools we use to analyse and predict various types of weather and climate extreme events. I will then focus on persistent climate extremes, like heatwaves and cold spells, and discuss the new insights we gain when analysing them through the lens of large deviations theory. This framework has given rise to the concept of "typicality" concerning certain weather and climate extremes, indicating their property to exhibit similarities in spatial patterns, temporal evolution, and underlying physical processes, with this resemblance intensifying as events become more extreme. Recent research confirms that highly intense heatwaves, characterized by prolonged local temperature anomalies, consistently coincide with specific large-scale circulation patterns. This suggests that there is a typical way the atmosphere realises an extreme temperature anomaly.

12:30–12:50 From Micro to Macro in Modeling Sea Ice

Kenneth Golden (University of Utah, USA)

Abstract Earth's sea ice covers form a key component of the climate system. Their precipitous declines impact the polar marine environment and its ecosystems, with ripple effects felt far beyond the polar regions. As a material sea ice exhibits composite structure on many length scales. A principal challenge is how to use microstructural information to find effective properties on larger scales relevant to climate and ecological models. From microscale brine inclusions in sea ice to the Arctic ice pack itself, and from algae living inside the brine inclusions to polar bears roaming the ice surface, we'll tour recent advances in modeling sea ice and the ecosystems it hosts. Areas of mathematics we'll visit include percolation, fractal geometry, random matrix theory, advection diffusion, inverse problems, topological data analysis, and uncertainty quantification.

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14:30–14:50 Analysis of a two-layer energy balance model: long time behaviour and greenhouse effect (Part 1)

Judith Vancostenoble (Université Toulouse 3, FR)

Abstract We study a two-layer energy balance model, that allows for vertical exchanges between a surface layer and the atmosphere. The evolution equations of the surface temperature and the atmospheric temperature are coupled by the emission of infrared radiation by one level, that emission being partly captured by the other layer, and the effect of all non-radiative vertical exchanges of energy. Therefore, an essential parameter is the absorptivity of the atmosphere, denoted ε_a. The value of ε_a depends critically on greenhouse gases: increasing concentrations of CO_2 and CH_4 lead to a more opaque atmosphere with higher values of ϵ_a.

First we study the associated ODE model, and we prove that global existence of solutions of the system holds if and only if $\varepsilon_a \in (0,2)$, and blow up in finite time occurs if $\varepsilon_a > 2$. (Note that the physical range of values for ε_a is $(0,1]$). Next, we explain the long time dynamics for $\varepsilon_a \in (0,2)$, and we prove that all solutions converge to some equilibrium point. We also present some numerical results showing the classical S-shaped structure of the equilibrium points. Finally, motivated by the physical context, we study the dependence of the equilibrium points with respect to the involved parameters, and we prove in particular that the surface temperature increases monotonically with respect to ε_a. This is the key mathematical manifestation of the greenhouse effect for this toy model.

Finally, we turn to the PDE model, and we prove global existence and convergence results, a key tool being the existence of comparison principles (and the informations obtained on the ODE model). We also provide some properties of the stationary solutions.

References

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15:00–15:20 Analysis of a two-layer energy balance model: long time behaviour and greenhouse effect (Part 2)

Patrick Martinez (Université Toulouse 3, FR)

Abstract We study a two-layer energy balance model, that allows for vertical exchanges between a surface layer and the atmosphere. The evolution equations of the surface temperature and the atmospheric temperature are coupled by the emission of infrared radiation by one level, that emission being partly captured by the other layer, and the effect of all non-radiative vertical exchanges of energy. Therefore, an essential parameter is the absorptivity of the atmosphere, denoted ε_a. The value of ε_a depends critically on greenhouse gases: increasing concentrations of CO_2 and CH_4 lead to a more opaque atmosphere with higher values of ϵ_a.

First we study the associated ODE model, and we prove that global existence of solutions of the system holds if and only if $\varepsilon_a \in (0,2)$, and blow up in finite time occurs if $\varepsilon_a > 2$. (Note that the physical range of values for ε_a is $(0,1]$). Next, we explain the long time dynamics for $\varepsilon_a \in (0,2)$, and we prove that all solutions converge to some equilibrium point. We also present some numerical results showing the classical S-shaped structure of the equilibrium points. Finally, motivated by the physical context, we study the dependence of the equilibrium points with respect to the involved parameters, and we prove in particular that the surface temperature increases monotonically with respect to ε_a. This is the key mathematical manifestation of the greenhouse effect for this toy model.

Finally, we turn to the PDE model, and we prove global existence and convergence results, a key tool being the existence of comparison principles (and the informations obtained on the ODE model). We also provide some properties of the stationary solutions.

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15:30–15:50 On response theory for climate models

Giulia Carigi (University of L'Aquila, IT)

Abstract A methodology to establish response theory for a class of nonlinear stochastic partial differential equations has been provided in [1]. Specifically, it is shown that for a certain class of observables, the averages of those observables against the stationary measure of the SPDE are differentiable (linear response) or, under weaker conditions, locally Hölder continuous (fractional response) as functions of a deterministic additive forcing.

The physical motivation for studying the response to perturbations in the forcings for models in geophysical fluid dynamics comes from climate change and relate to the question as to whether long term statistical properties of the dynamics derived under current conditions will be valid under different forcing scenarios.

In this talk we look at the the key features of the methodology for stochastic PDEs in [1], with a particular interest on its application to quasi-geostrophic models for the ocean and energy balance models for the global temperature.

References

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16:00–16:20 Linear response and control of the statistical properties of dynamics

Stefano Galatolo (Università di Pisa, IT)

Abstract. The concept of linear response of a dynamical system describes how the statistical properties of the system change when a certain "infinitesimal" perturbation is applied to the

system. This concept has important relations with the study of climate models and the climate change ([5]).

One inverse problem associated to linear response is the one of finding the best infinitesimal perturbation in order to modify the statistical properties of the system in some wanted direction. This inverse problem can be formalized in different ways (see e.g. [1],[2],[3],[5]). The existence of an optimal solution has been proved in some classes of deterministic and random systems and numerical methods for its approximation have been shown. In the talk we will briefly discuss the mathematical structure of the problem and review some recent result on the control of the statistical properties of random dynamical systems and expanding maps.

References

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17:00–17:20 The Geometry of Time-Dependent Spherical Random Fields
Anna Vidotto (University of Naples Federico II, IT)

Abstract. Space-time random fields defined over the two-dimensional unit sphere $\mathbb{S}^2$ find a wide set of applications in Climate Sciences. Moreover, the investigation of the behavior for geometric functionals of random fields on manifolds has drawn recently considerable attention. In this talk, we consider fluctuations over time for the excursion area and the level curves' length of time-dependent Gaussian spherical random fields. We focus on both long and short memory assumptions, presenting a different behavior starting from the analysis of the Wiener chaos decompositions of the geometric functionals of interest. In particular, in the long memory case, we show that the fluctuations are dominated by a single chaotic component, while in the short memory case, we show that all chaoses contribute in the limit, no variance cancellation occurs and a Central Limit Theorem can be established by Fourth-Moment Theorems and a Breuer-Major argument. The talk is based on the two articles [1, 2].

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17:30–17:50 Topological analysis of weather extremes
Gisela Daniela Charó (Laboratoire des Sciences du Climat et de l'Environnement, Université Paris-Saclay & IPSL, FR; CONICET – Universidad de Buenos Aires. Centro de Investigaciones del Mar y la Atmósfera (CIMA), AR)

Abstract Volumes in a dynamical system's phase space can stretch, compress, fold or tear; the combination of these processes gives rise to a structure in its space. The topology of this structure is the signature of the mechanisms that act to shape the system's flow in this phase space [1]. Our topological approach uses the novel concept of templex [2], which consists of finding a topological representation for the branched manifold approximated by a cell complex and a directed graph that prescribes the allowed connections between the highest dimensional cells of the complex as a function of the flow. The templex properties include the non-equivalent ways of circulating along

the complex, which are essential to provide a complete description of the system's dynamics. The templex provides the key features of the topological structure underlying a dynamical system.

Weather extremes can be classified by two scalars: the instantaneous dimension and the persistence of a state of a dynamical system [3,4]. The computation of these asymptotic scalars for weather observables provides information on the rarity, and persistence of specific states. In this work, we present a first attempt to relate the instantaneous dimension and other local metrics to the topological properties of the templex of the system under study.

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18:00–18:20 Dynamical units for attractors with high-order symmetries extracted from templex

Christophe Letellier (Rouen Normandy University, FR)

Abstract Chaotic attractors produced by three-dimensional systems are well characterized by a template that can be viewed as a knot-holder: consequently, the approach was mainly based on knot theory [1,2,3]. Unfortunately, in higher dimensional space, all knot are trivial and this theory can no longer be used. Recently, we introduced a new concept, templex, based on the construction of a complex of cells from a trajectory visiting the chaotic attractor [4,5]. The key novelty was i) to orientate the cell according to the flow, ii) to associate with such a complex the flow which governs the evolution of the trajectory, thus leading to a directed graph which synthetizes most of the topological properties of the attractor. Since the complex can be constructed in any dimension, a templex can be obtained for high-dimensional attractor. By reducing the complex to a *minimal* form, the corresponding reduced digraph allows to extract some dynamical units which are of two types, oscillating (O) and switching (S), and which structure the attractor. Thus, the attractor is viewed as a combination of units based on which a taxonomy of chaotic invariant sets, initially introduced by Otto E. Rössler [6] and recently extended [7], can be generalized. Chaotic attractors with high-order symmetries [8,9] are here investigated in terms of templexes and their structure in terms of O- and S-units are provided.

A climate attractor is treated with this procedure validated with the previous examples.

References

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July 26, 2024

11:30–11:50 Acoustic waves analysis in atmosphere modelling
Donatella Donatelli (Università degli Studi dell'Aquila, IT)
Abstract Fluid dynamic equations are used to model various phenomena arising from physics, engineering, astrophysics, geophysics. In particular in the case of a geophysical flow, such as the atmospheric flows one feature is that they take place at different time and length scales and it is important to understand which phenomena occur according to the use of single scales or to the interactions of them (i.e. internal gravity waves, Rossby waves, cloud formation). From a mathematical point of view, these various physical behaviors give rise to different singular limits and, consequently to a different analysis of the asymphtotics of the governing equations. In this talk we will focus on a combined low Mach number, low Rossby number limit and we will analyze a very simplified model given by a linearized continuity equation and by the classical momentum equation which include terms that take into account of gravitation and rotation and we will show, according to the values of different scales, that the asymptotic behavior of the model will be those of an incompressible fluid or of a geostrophic flow (see [1]).

The common feature of this kind of limit in the ill prepared data framework is, the presence of high frequency time oscillations along the so called acoustic waves. Those waves are supported by the gradient part of the velocity field, which, as consequence becomes infinte. In this scenario it becomes extremely important to understand under which conditions the behaviour of this waves can be controlled and their presence is harmless. In this talk we will show in which cases those waves have a dispersive behaviour and how dispersion may help in order to recover the necessary compactness of the velocity field.

References

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12:00–12:20 A quasi-geostrophic coupled ocean atmosphere model
Tobias Kuna (DISIM, University of L'Aquila and Center for Mathematics of Planet Earth, IT)
Abstract Ocean atmosphere coupling and thermodynamical effects are essential to understand low frequency variability on annual and decadal scale. We study well-posedness of a system of PDEs describing the atmosphere by two quasi-geostrophic layers coupled to one further quasi-geostrophic deep ocean layer. Furthermore, there are two equations describing the development of the atmosphere and ocean temperature. More specifically, we consider the model which Vannitsem et al. in [1] used, which is based on previous models by Charney and Strauss '80, Reinhold and Pierrehumbert '82; Pierini '11. These models have been studied intensively numerically, but not analytically. We will describe the first steps in this direction: existence of weak and strong solutions, weak-uniqueness, existence and finite-dimensionality of the global attractor, determining modes. An interesting asymmetry in the unknowns with respect to regularity, dimension of attractor and determining modes emerges.

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12:30–12:50 Dynamical systems approaches to the study of climate extremes with applications to recent events

Davide Faranda (CNRS, FR)

Abstract Between 1980 and now, climate and extreme weather events have caused, in Europe, up to 520 billion euros in economic losses, according to a 2022 report released by the European Environment Agency. Many weather extreme events such as heatwaves, droughts, tropical cyclones and cold spells will change in frequency and/or intensity due to the ongoing anthropogenic climate change, driving our planet to unprecedented threats. From a mathematical point of view, such extreme events are difficult to characterize because they are rare and they do occur at specific spatiotemporal scales of the dynamics, not necessarily the largest or the smallest of the climate system. Their study requires a dynamical systems framework capable of tracking their probability of occurrence, predictability and persistence. In this talk I will describe how dynamical systems theory helps in building numerical and theoretical tools for weather extreme events, using recurrences of patterns termed analogues. These tools can be used to study whether climate change do already affect the dynamics of extremes.

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14:30–14:50 The interaction of physics and mathematics: modelling, solving and simulating large-scale flows

Beatrice Pelloni (Heriot-Watt University, UK)

Abstract This talk focuses on one particular model, the semi-geostrophic equations . First proposed in 1948 by meteorologist Arnt Eliassen, the semi-geostrophic equations constitute a model for the formation of fronts in the atmosphere. While these equations have been well studied by the meteorological community, fundamental problems concerning their mathematical analysis remain open. In this talk, I will present a novel constructive proof of the existence of *geostrophic* Lagrangian solutions using semi-discrete optimal transport techniques, for both incompressible and compressible flows. This proof gives rise to a very effective numerical method, in the full 3D setting. It might also offer a pathway to settle the existence and uniqueness of Eulerian solutions, thus addressing a major theoretical open problem.

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15:00–15:20 Reconstruction of forcings models of geophysical fluid dynamics

Jochen Bröcker (University of Reading, UK)

Abstract In this contribution, we consider data assimilation schemes that not only estimate the underlying states of a dynamical system but simultaneously reconstruct unknown components of the dynamics.

We focus on simple 2D-Navier Stokes and Transport-Diffusion equations (for instance for atmospheric aerosols or tracer gases) and reconstruct forcings or surfacd fluxes, along with the underlying dynamical states. Tracer gases and aerosols play an important role in the dynamics of the atmosphere; tracer gases such as ozone, methane, or CO_2 for instance impact the radiative transfer and are thus linked to the energy budget of the planet ("greenhouse effect"), while aerosols

(especially in the lower troposphere) are common pollutants with strong and potentially adverse effects on the environment, human activity, and health.

We discuss two algorithms that both apply in the context of the 2D-Navier Stokes as well as the transport–diffusion equations. The discussed algorithms are very simple and not optimised for performance, but they permit a fully rigorous mathematical analysis. Insights from this analysis can thus inform the design of more complex (and better performing) approaches.

15:30–15:50 Background states of the atmosphere: analysis and computation using Optimal Transport

Charlie Egan (University of Göttingen, DE)

Abstract The decomposition of an atmospheric flow into a background state and perturbations to it plays a central role in several areas of atmospheric science. This talk will describe how the theory of Optimal Transport (OT) can be used to uniquely define and reliably compute such a background state. Specifically, we consider energy minimisers over the class of *Modified Lagrangian Mean* (MLM) states proposed in [1]. Translated in to the language of OT, such states can be viewed as minimisers of a suitable *optimal transport cost*, over a class of *source measures*, each one defined by a free surface. Using this viewpoint, we prove the existence and uniqueness of such states. In fact, optimality conditions imply that this free surface optimisation problem reduces to a standard OT problem, whose solution implicitly defines the energy minimising free surface. Thanks to recent advances in numerical OT, the computation of energy minimising MLM states is therefore numerically tractable.

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16:00–16:20 Filtering Dynamical Systems Using Observations of Statistics

Eviatar Bach (University of Reading, UK)

Abstract We consider the problem of filtering dynamical systems, possibly stochastic, using observations of statistics [1]. Thus, the computational task is to estimate a time-evolving density $\rho(v,t)$ given noisy observations of the true density $\rho^\dagger$; this contrasts with the standard filtering problem based on observations of the state v. The task is naturally formulated as an infinite-dimensional filtering problem in the space of densities ρ. However, for the purposes of tractability, we seek algorithms in state space; specifically, we introduce a mean-field state-space model, and using interacting particle system approximations to this model, we propose an ensemble method. We refer to the resulting methodology as the ensemble Fokker–Planck filter (EnFPF).

Under certain restrictive assumptions, we show that the EnFPF approximates the Kalman–Bucy filter for the Fokker–Planck equation, which is the exact solution to the infinite-dimensional filtering problem. Furthermore, our numerical experiments show that the methodology is useful beyond this restrictive setting. Specifically, the experiments show that the EnFPF is able to correct ensemble statistics, to accelerate convergence to the invariant density for autonomous systems, and to accelerate convergence to time-dependent invariant densities for non-autonomous systems. We discuss possible applications of the EnFPF to climate ensembles and to turbulence modeling.

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17:00–17:20 Galerkin Approximations of Nonlinear DDEs: Bifurcation Analysis and Noise-Driven Chaos

Honghu Liu (Virginia Tech, USA)

Abstract Delay differential equations (DDEs) are widely used in many applied fields to account for delayed responses of the modeled systems to internal/external factors. In contrast to ODEs, the

phase space associated even with a scalar DDE is infinite-dimensional. Oftentimes, it is desirable to have low-dimensional ODE systems to approximate the DDE dynamics. In this talk, we will discuss a recently developed Galerkin scheme for general nonlinear DDEs. The main ingredient is a type of polynomials that are orthogonal under an inner product with a point mass. The associated Galerkin scheme enjoys some nice properties that help reduce the derivation of the convergence results to basic functional analysis exercises. Analytic formulas are available within this approach, which help simplify the numerical treatment.

We will also discuss further dimension reduction using the center manifold technique for DDE bifurcation analysis. The developed framework leads to a rigorous and computationally efficient way to approximate the Stuart-Landau normal forms for the considered DDEs. We will show how insights gained from such normal forms can help design stochastic perturbations to further enrich the time variability of the otherwise periodic DDE dynamics. The approach will be illustrated on a cloud-rain DDE model in the context of Hopf bifurcations and noise-driven chaos, and also on a delay model of the El Niño-Southern Oscillation (ENSO) in the context of saddle-node bifurcation of periodic orbits and noise induced decadal variability.

References

[1] M. D. Chekroun, M. Ghil, H. Liu, S. Wang, *Low-dimensional Galerkin approximations of nonlinear delay differential equations*, Discrete Continuous Dyn. Syst. 36 (2016), 4133–4177.

[2] M. D. Chekroun, I. Koren, H. Liu, *Efficient reduction for diagnosing Hopf bifurcation in delay differential systems: Applications to cloud-rain models*, Chaos 30 (2020), 053130.

[3] M. D. Chekroun, I. Koren, Honghu Liu, Huan Liu, *Generic generation of noise-driven chaos in stochastic time delay systems: Bridging the gap with high-end simulations*, Science Advances, 8 (2022), eabq7137.

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17:30–17:50 A pollution model on Network
Marta Leocata (LUISS, IT)

Abstract We study the joint determination of optimal investment and optimal depollution in a spatiotemporal framework where pollution is transboundary and spatial component is described by a network structure. Pollution is controlled at a global level by a central planner. The problem is solved explicitly and the optimal investment and depollution are found. In conclusione, some investigations on the impact of heterogeneity on the optimal path are performed.

18:00–18:20 Analysis and control for degenerate evolution equations with applications to climate sciences
Giuseppe Floridia (Sapienza University of Rome, IT)

Abstract In this talk we study the Energy Balance Climate Models (EBCM), in particular we consider the Budyko-Sellers model, that is a particular kind of EBCM.

We talk about some results concerning the approximate multiplicative controllability of degenerate reaction-diffusion equations with applications to the 1-D Budyko-Sellers model.

We also introduce a new version of the Budyko-Sellers model using time-fractional degenerate heat equations to simulate anomalous heat diffusion in several situations.

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