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# Recent developments in commutative algebra
# Special Session B20

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Commutative algebra is the field of mathematics that studies commutative rings and their ideals and modules. The subject has deep connections with other fields including algebraic geometry, combinatorics, representation theory, and algebraic topology. In the past few years, there have been several important developments in the field, leading to the solution to many long-standing conjectures and the introduction of new methods.

With this special session, we intend to integrate senior and young active researchers in the field to present their most recent advances and to facilitate the creation of new collaborations. The focus will be on recent progress in the topics of free resolutions, local cohomology, singularities, positive characteristic methods, and interactions of commutative algebra with combinatorics.

For more information visit `https://sites.google.com/view/ca2024palermo/home`.

**Schedule and Abstracts**

July 25, 2024

**11:30-12:15 Cohomology of line bundles on the incidence correspondence**
**Claudiu Raicu (University of Notre Dame, USA)**
*Abstract.* A fundamental problem at the confluence of algebraic geometry, commutative algebra and representation theory is to understand the structure and vanishing behavior of the cohomology of line bundles on (partial) flag varieties. Over fields of characteristic zero, this is the content of the Borel–Weil–Bott theorem and is well-understood, but in positive characteristic it remains wide open, despite important progress over the years. In my talk I will describe recent developments obtained over the past couple of years in the case of the incidence correspondence – the partial flag variety consisting of pairs of a point in projective space and a hyperplane containing it.

**12:30-12:50 Standard monomial theory modulo Frobenius in characteristic two**
**Laura Casabella (MPI for Mathematics in the Sciences, Leipzig, Germany)**
*Abstract.* Over a field of characteristic zero, De Concini, Eisenbud and Procesi developed a theory of standard monomials, which are a vector space basis for determinantal ideals and provide a tool to study many properties of the ideal. This theory plays a central role in the study of determinantal rings from a representation theoretic approach, and exploits tableaux combinatorics.

In this talk, we present our contribution to a new standard monomial theory for polynomial rings over a field of positive characteristics modulo a Frobenius power, examining the characteristic two case. A main feature of this investigation is given by analogs of semistandard Young tableaux and Schur polynomials in this new context, defined by Gao, Raicu and VandeBogert. Our results agree with one of their conjectures.

**References**

[1] L. Casabella, T. Yu, *Standard monomial theory modulo Frobenius in characteristic two*, (2023) `arXiv:2311.05530`.

[2] Z. Gao, C. Raicu, and K. VandeBogert, *Some questions arising from the study of cohomology on flag varieties*, (2023) `arXiv:2308.04334`.

[3] C. De Concini, D. Eisenbud, and C. Procesi, *Young diagrams and determinantal varieties*, Invent. Math. 5 (1980), no. 2, 129-165.

**14:30-15:15 Resolutions of plane curve singularities and Conway–Coxeter friezes**
**Eleonore Faber (Universität Graz, Austria)**
*Abstract.* Conway–Coxeter friezes are arrays of positive integers satisfying a determinantal condition, the so-called diamond rule. Recently, these combinatorial objects have been of considerable interest in algebra, since they encode cluster combinatorics of type A. In this talk I will discuss a new connection between Conway–Coxeter friezes and the combinatorics of a resolution of a plane curve singularity: via the beautiful relation between friezes and triangulations of polygons one can relate each frieze to the so-called lotus of a curve singularity, which was introduced by Popescu-Pampu. This allows to interpret the entries in the frieze in terms of invariants of the curve singularity, and on the other hand, we can see cluster mutations in terms of the desingularization of the curve.

**15:30-15:50 Generic Distractions and Initial Ideals**
**Giulio Caviglia (Purdue University, USA)**
*Abstract.* Given a homogeneous ideal $I$ in a polynomial ring $S$ over a field and a monomial order, we iteratively compute initial ideals and ad-hoc monomial distractions in order to construct a distinguished monomial ideal of $S$ associated to $I$. We call it the distraction-generic initial ideal of $I$ and denote it by $D$-gin($I$). Such an ideal, when char($K$)=0, agrees with the usual generic initial ideal of $I$. Furthermore, it is strongly stable in any characteristic and, when computed with respect to the reverse lexicographic order, it has properties analogous to gin($I$), for instance, it has the same Castelnuovo-Mumford regularity and projective dimension as $I$. As an application, we can use it to extend to $\mathrm{char}(K) = p$ some work done by Mall in the nineties on strata of the Hilbert schemes defined by fixing a Hilbert series and an upper bound for the Castelnuovo-Mumford regularity.

This is a joint work with Anna-Rose Wolff.

**16:00-16:20 Componentwise linearity under Gröbner degenerations**
**Hongmiao Yu (University of Genoa, Italy)**
*Abstract.* In this talk, we will discuss when Gröbner degenerations preserve the componentwise linearity. More concretely, given that $R = K[X_1, \dots, X_n]$ is the polynomial ring over a field $K$ and $I$ is a homogeneous ideal of $R$, a theorem of Conca and Varbaro implies that if the initial ideal in($I$) is square-free, then $I$ has linear resolution if and only if in($I$) has linear resolution. Since each homogeneous ideal with linear resolution is componentwise linear, we aim to explore the conditions under which the property of being componentwise linear can be transferred between $I$ and in($I$). Additionally, we will compare the graded Betti numbers of $I$ and in($I$) under these conditions.

**17:00-17:20 Source-independence of generalized F-signature**
**Ilya Smirnov (BCAM-Basque Center for Applied Mathematics, Ikerbasque, Basque Foundation for Science)**
*Abstract.* Let $R$ be an commutative Noetherian domain containing the finite field $\mathbb{F}_p$. We assume that the Frobenius endomorphism $F\colon R \to R$, $x \mapsto x^p$, is a finite map. This assumption is satisfied if $R$ is a complete local ring with a perfect residue field or if $R$ is localization of a finite type algebra over a perfect field. With this assumption, for any finite $R$-module $M$ the module $F^e_* M$, obtained by restricting the scalars along the $e$th iterate of the Frobenius, is still finitely generated.

**Definition 1** (Sannai, Huneke – Leuschke, Smith – Van der Bergh)**.** Let $Q$ be the fraction field of $R$. The dual F-signature of a finitely generated $R$-module $M$ is defined as

$$s_{\mathrm{dual}}(M) = \lim_{e\to\infty} \frac{\max\{n \mid \text{there is a surjection } F^e_* M \to \oplus^n M \to 0\}}{\dim_Q(Q \otimes_R F^e_* M)}.$$

In the case $M = R$ this recovers the definition of F-signature introduced by Huneke–Leuschke. Previously, the existence of the limit was established in [4] for $M = R$ and in [3] for $M = \omega_R$, the dualizing module of a Cohen-Macaulay ring. Using the linear algebra approach to building surjections established in [3] we prove that the limit exists in full generality for more general type of limits.

**Theorem 1.** *For any finitely generated $R$-modules $M, N$*

$$s_{\mathrm{dual}}(M) = \lim_{e\to\infty} \frac{\max\{n \mid \textit{there is a surjection } F^e_* N \to \oplus^n M \to 0\}}{\dim_Q(Q \otimes_R F^e_* N)}$$

*and the limit exists.*

The fact that one may compute $s_{\mathrm{dual}}(M)$ from any $N$ is especially useful when $R$ is an invariant subring under an action of a finite group $G$ on a polynomial ring $S = k[x_1, \ldots, x_d]$ or, more generally, when there is a finite map $R \to S$ where $S$ is a regular local ring. In this case, $F^e_* S$ is a free $S$-module by a celebrated theorem of Kunz.

**Corollary 2.** *Suppose that $S$ is a regular local ring and $R \to S$ is a finite map. Then $s_{\mathrm{dual}}(M) > 0$ if and only if there exists a surjection of $R$-modules $\oplus^n S \to M \to 0$ for some $n > 0$.*

We recover a result of Hashimoto obtained by representation theory in [1].

**Corollary 3.** *Suppose that $S = k[x_1, \ldots, x_d]$ is $R = S^G$ where $G$ is a finite subgroup containing no pseudo-reflections. Suppose that $R$ is Cohen-Macaulay and has a dualizing module $\omega_R$. Then the following are equivalent:*

(1) *$R$ is F-rational,*
(2) $s_{\mathrm{dual}}(\omega_R) > 0$*,*
(3) $s_{\mathrm{dual}}(\omega_R) \geq 1/|G|$*,*
(4) *there exists a surjection of $R$-modules $S \to \omega_R \to 0$.*

**References**


[1] M. Hashimoto, *F-rationality of the ring of modular invariants*, J. Algebra 484, 207–223, 2017.

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**17:30-17:50 F-purity of ladder determinantal varieties and their symbolic blowups**
**Lisa Seccia (University of Neuchâtel, Switzerland)**
*Abstract.* Ladder determinantal varieties are defined by the vanishing of minors in certain subsets of a generic matrix of indeterminates. These varieties were initially introduced to investigate singularities of Schubert varieties, and have since inspired further study due to their rich algebraic structure.

In this talk, we focus on $F$-singularities of ladder determinantal varieties and their symbolic blowups. After a brief overview of some basic concepts of $F$-singularity theory, we will present different algebraic techniques to prove that these varieties are $F$-pure. Time permitting, we will discuss a way to extend these techniques to other classes of ideals.

**18:00-18:20 Variants of the Buchsbaum property in prime characteristic**
**Austyn Simpson (University of Michigan, USA)**
*Abstract.* I will discuss some recent refinements of the notion of a Buchsbaum ring in prime characteristic. Recall that a local ring $R$ is said to be *Buchsbaum* if the quantity $e(Q) - \ell_R(R/Q)$ does not depend on the parameter ideal $Q$, where $e(Q)$ denotes the multiplicity. Recently, Ma

and Quy considered a prime characteristic alternative requiring instead for $e(Q) - \ell_R(R/Q^*)$ to be independent of parameter ideals $Q$, where $Q^*$ is the tight closure of $Q$.

I will report on some ongoing investigations into this "tight Buchsbaum" notion such as various homological interpretations, its behavior under a section, and a characterization in terms of Rees algebras. Time permitting, I will also discuss yet another alternative which replaces $Q^*$ with the Frobenius closure $Q^F$ in the above quantity.

This is joint work with A. Costantini, K. Goel, K. Maddox, and L.E. Miller.

July 26, 2024

**11:30-12:15 Licci ideals**

**Claudia Polini (University of Notre Dame, USA)**

*Abstract.* Linkage was introduced in the nineteenth century as a method to classify projective varieties. Using linkage we can define an equivalence relation and an ideal is called **licci** if it is in the linkage class of a complete intersection. Since linkage preserves many properties of an ideal, licci ideals are particularly nice. Standard examples of licci ideals include perfect ideals of grade 2 and perfect Gorenstein ideals of grade 3. Goals of the subject are to classify linkage classes, to establish properties of licci ideals, to find new classes of licci ideals, and to find necessary and sufficient conditions for ideals to be licci. In this talk I will discuss a surprising conjecture: *licci ideals do not have many minimal generators*! Indeed in collaboration with Huneke and Ulrich we conjectured that the number of generators of a homogeneous licci ideal is bounded above by the greatest last twist in a minimal graded free resolution of the ideal. We proved this conjecture for several classes of ideals, for instance for monomial ideals of finite colenght, ideals containing a maximal regular sequence of quadrics, and licci ideals with nearly pure resolutions. Moreover, we give a sufficient condition for an ideal containing a maximal regular sequence of quadrics to be licci.

**12:30-12:50 Jet schemes of Pfaffian ideals**

**Emanuela De Negri (University of Genoa, Italy)**

*Abstract.* Jet schemes and arc spaces received quite a lot of attention by researchers after their introduction, due to J. Nash, and established their importance as an object of study in M. Kontsevich's motivic integration theory. Several results point out that jet schemes carry a rich amount of geometrical information about the original object they stem from, whereas, from an algebraic point of view, little is know about them.

In this talk we consider the ideal $I_{2r}$ generated by the pfaffians of size $2r$ in an $n \times n$ generic skew-symmetric matrix and, inspired by [2], we study algebraic properties of the corresponding $k$-th jet schemes ideal $I_r^{n,k}$. In particular we determine under which conditions the corresponding jet scheme varieties are irreducible. Moreover in the case $n = 2r$ we prove that for every $k$ the natural generators of $I_r^{n,k}$ are a Gröbner basis, and that $I_r^{n,k}$ defines a Cohen Macaulay domain of multiplicity $r^k$. Conjectures and open questions will be stated.

**References**

[1] E. De Negri, E.Sbarra, *On jet schemes of pfaffian ideals*, Collect. Math., 70 (2019), 479–491.

[2] T. Košir B.A Sethuraman, *Determinantal varieties over truncated polynomial rings*, J. Pure Appl. Algebra, 195 (2005), 75–95.

**14:30-15:15 Lefschetz properties for artinian Gorenstein algebras of low Sperner number**

**Mats Boij (KTH Royal Institute of Technology, Sweden)**

*Abstract.* The cohomology ring $A$ of a smooth projective complex variety has the *strong Lefschetz property* (SLP), i.e., all the multiplication maps, $\times\ell^j\colon A_i \longrightarrow A_{i+j}$, given by powers of a general linear forms have maximal rank. For other artinian Gorenstein algebras in general, i.e., commutative algebras with Poincaré duality, it is well known that this needs not be true, not even for multiplication by general linear forms, $\times\ell\colon A_i \longrightarrow A_{i+1}$, which would be the *weak*

*Lefschetz property* (WLP). However, there are lots of results and some conjectures about when we have the strong or the weak Lefschetz property. In codimension three, we know that artinian Gorenstein algebras satisfy the SLP if the socle degree is at most 5 by [3] and if the *Sperner number*, i.e., the maximal value of the Hilbert function, is at most 6 by [1]. It is also known that artinian complete intersections satisfy the WLP in codimension three [4]. For Gorenstein algebras in codimension four, the smallest known examples of artinian Gorenstein algebras have Sperner number equal to the socle degree plus two [2].

In recent joint work we prove that any artinian Gorenstein algebras with socle degree $d$ and Sperner number at most $d+1$ satisfies the WLP.

**References**


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[2] Mats Boij. Components of the space parametrizing graded Gorenstein Artin algebras with a given Hilbert function. *Pacific J. Math.*, 187(1):1–11, 1999.

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[4] Tadahito Harima, Juan C. Migliore, Uwe Nagel, and Junzo Watanabe. The weak and strong Lefschetz properties for Artinian $K$-algebras. *J. Algebra*, 262(1):99–126, 2003.


**15:30-15:50 Lattice Paths, Lefschetz Properties and Almkvist's Conjecture in Two Variables**

**Nancy Abdallah (University of Borås, Sweden)**

*Abstract.* Fix a field $F$, and positive integers $m, n$. Define the algebra

$$A_F(m,n) = \frac{F[e_1,\ldots,e_n]}{(e_1(m),\ldots,e_n(m))},$$

where $e_i = e_i(x_1,\ldots,x_n) = \sum_{1\le j_1<\cdots<j_i\le n} x_{j_1}\cdots x_{j_n}$ is the $i^{th}$ elementary symmetric function, and $e_i(m) = e_i(x_1^m,\ldots,x_n^m)$. $A(m,n)$ is a graded Artinian complete intersection. In [1], Almkvist conjectures that the Hilbert function of $A(m,n)$ is unimodal for $n$ odd and sufficiently large $m$, and for $n$ even and any $m$. Since unimodality is a necessary condition for Lefschetz properties, we conjecture that $A(m,n)$ has the strong Lefschetz property for sufficiently large $m$, and for even $n$. In this talk, we consider the case $n=2$, and we show that $A(m,2)$ has the strong Lefschetz property and the complex Hodge-Riemann property if and only if $m$ is even.

**References**


[1] G. Almkvist, *Proof of a conjecture about unimodal polynomials,* J. Number Theory 32 (1989), 43–57.


**16:00-16:20 On the (strong) Koszul property for some Artinian Gorenstein algebras**

**Alessio D'Alì (Politecnico di Milano, Italy)**

*Abstract.* A standard graded commutative $K$-algebra $R$ is *Koszul* when the residue field $K$ has a linear free resolution as an $R$-module; in particular, the defining ideal of $R$ has to be quadratic. Koszulness is a natural feature of many algebras arising in combinatorics and algebraic geometry. In some cases, an even stronger property holds: namely, there exists a $K$-basis $\mathfrak{B}$ of $R_1$ such that, for every choice of a proper subset $\mathfrak{B}' \subsetneq \mathfrak{B}$ and of an element $x \in \mathfrak{B} \setminus \mathfrak{B}'$, the colon ideal $(y \mid y \in \mathfrak{B}') :_R (x)$ is generated by a subset of $\mathfrak{B}$. Standard graded $K$-algebras $R$ with this property were called *strongly Koszul* by Herzog, Hibi and Restuccia.

In this talk I will discuss Koszulness for some classes of Artinian Gorenstein rings, possibly including some ongoing work about the strong Koszul property for some Artinian Gorenstein rings related to determinantal objects and to secant varieties of Severi varieties.

**References**

[1] J. Herzog, T. Hibi, G. Restuccia, *Strongly Koszul algebras*, Math. Scand., 86 (2000), 161–178.

**17:00-17:20 Commutative algebra of polyominoes**
**Francesco Romeo (University of Cassino and Southern Lazio, Italy)**
*Abstract.* Polyominoes are two-dimensional objects obtained by joining edge by edge squares of same size. Originally, polyominoes appeared in mathematical recreations, but it turned out that they have applications in various fields, for example, theoretical physics and bio-informatics. Among the most popular topics in combinatorics related to polyominoes one finds enumerating polyominoes of given size, including the asymptotic growth of the numbers of polyominoes, tiling problems, and reconstruction of polyominoes. Recently Qureshi [5] introduced a binomial ideal induced by the geometry of a given polyomino, called polyomino ideal, and its related algebra. From that moment different authors studied algebraic properties and invariants related to this ideal (see [3, 6, 4, 2, 7, 1]). In this talk, we provide a comprehensive overview of the state-of-the-art results that have been obtained on polyomino ideals and their related algebra. In the first part of the talk, we discuss questions about the primality and the Gröbner bases of the polyomino ideal. In the second part, we talk over some algebraic invariants such as Castelnuovo-Mumford regularity, Hilbert series, and Gorensteinness and related properties of the polyomino ideal and its coordinate ring.

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**17:30-17:50 Binomial edge ideals and Serre's condition (S2)**
**Francesco Strazzanti (Università di Genova, Italy)**
*Abstract.* A binomial edge ideal is an ideal of a polynomial ring, generated by binomials corresponding to the edges of a finite simple graph. It can be also viewed as the ideal generated by *some* minors of a generic matrix with two rows.

The aim of this talk is to point out the connections between the combinatorics of finite simple graphs and the algebraic properties of binomial edge ideals. In particular, after reviewing some basic results, I will focus on a combinatorial characterization of binomial edge ideals satisfying Serre's condition $(S_2)$.

**References**

[1] D. Bolognini, A. Macchia, G. Rinaldo, F. Strazzanti, *A combinatorial characterization of $S_2$ binomial edge ideals*, arXiv:2306.17076.

**18:00-18:20 A constructive approach for graphs whose binomial edge ideal is Cohen-Macaulay**
**Giancarlo Rinaldo (Università di Messina, Italy)**
*Abstract.* Binomial edge ideals have been introduced in [5] and, independently, in [7]. They are associated to finite simple graphs, in fact they arise from the 2-minors of a $2 \times n$ matrix

related to the edges of a graph with $n$ vertices. The problem of finding a characterization of Cohen–Macaulay binomial edge ideals has been studied intensively by many authors (e.g. [4],[1]). In this talk we present a computational approach to find (see [6]) or construct graphs (see [2]) whose binomial edge ideal is Cohen-Macaulay by a library of the author. Thanks to this computation we obtained graphs with nice properties within the Cohen-Macaulay ones. Moreover, we verify a recent conjecture of D. Bolognini, A. Macchia and F. Strazzanti (see [3]).

**References**


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