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Graphs Associated with Groups: Advances and Applications. Special Session A7

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This special session aims to bring together mathematicians interested in exploring the synergy between graphs and group theory. Graph theory and group theory are two fundamental branches of mathematics with a wide range of applications in various fields. In particular, the situation becomes quite interesting when working in the intersection of these two areas, where a variety of ideas belonging to the first theory can be applied to the other. In recent years, the application of graphs to the study of groups and their properties has gained significant attention, as documented by several inspiring papers in literature. The session will provide a platform to discuss recent developments, share research findings, and foster collaboration in this exciting interdisciplinary field. In fact we seeks to promote the exchange of ideas, techniques, and applications of graph theory in the context of group theory by facilitating discussions on recent advancements, open problems and potential directions for future research. Participants will gain insights into how graph theory can be used to solve problems in group theory and vice versa.

For more information visit `https://sites.google.com/unisa.it/umi-ams-a7/home`.

Schedule and Abstracts

July 23, 2024

11:00–11:45 The total graph of a gain graph
Alfredo Donno (Niccolò Cusano University of Rome, ITALY)

Abstract. The *total graph* $T(\Gamma)$ of a graph $\Gamma = (V_\Gamma, E_\Gamma)$ was introduced by M. Behzad in his doctoral thesis in 1965. It is the graph with vertex set $V_\Gamma \cup E_\Gamma$, whose adjacencies are such that:

- $u \sim v$ in $T(\Gamma)$ when $u, v \in V_\Gamma$ and $u \sim v$ in Γ;
- $v \sim e$ in $T(\Gamma)$ when $v \in V_\Gamma$ is an endpoint of $e \in E_\Gamma$;
- $e \sim f$ in $T(\Gamma)$ when $e, f \in E_\Gamma$ are incident in Γ.

In particular, the graph Γ and its line graph are two subgraphs of $T(\Gamma)$. The spectrum of the total graph of a regular graph was investigated by Cvetković in 1973.

Recently, two possible definitions were introduced for the total graph of a signed graph: a signed graph (Γ, σ) consists of an underlying graph Γ, endowed with a signature function σ associating a sign ± 1 with each edge.

A further generalization of a signed graph is given by the notion of *gain graph*: a gain graph over a group G is a pair (Γ, ψ) consisting of an underlying graph Γ, endowed with a gain function ψ associating an element of a group G with every oriented edge, in such a way that the inverse element is associated with the opposite orientation of that edge.

We present here a definition of total graph for a gain graph by using G-phases, which can be regarded as the generalization of the notion of incidence matrix. Our construction is well defined, in the sense that gain graphs that are switching equivalent have switching equivalent total graphs; moreover, it recovers the existing notions of total graph for signed graphs.

By defining a suitable action of the gain group G on the set of G-phases, we are able to describe the sets of G-phases inducing the same switching equivalence class of gain functions on Γ or, equivalently, on its total graph.

Finally, we determine the spectrum of the total graph of a gain graph (Γ, ψ) whose underlying graph is regular: we express its eigenvalues in terms of the eigenvalues of (Γ, ψ), where the notion of spectrum is referred to a unitary representation of the gain group G.

12:00–12:45 Groups whose common divisor graph on p-regular classes is a forest
Víctor Sotomayor (Universitat Politècnica de València, SPAIN)

Abstract. Let G be a finite p-separable group, where p is a prime. In the last decades, some authors have investigated how the features of conjugacy classes of non-central p-regular elements of G inclucence its local structure. In this framework, the common divisor graph built on the mentioned set of classes has proven to be a very useful tool to capture certain arithmetical properties of them, which constrain the structure of the p-complements of G. The aim of this talk is to present new progress in this research area. More concretely, we will classify the structure of the p-complements of G when the common divisor graph on conjugacy classes of non-central p-regular elements is a forest. In particular, we will provide further evidence for some open problems regarding this graph.

14:30–15:15 Groups and Graphs: recent results
Mariagrazia Bianchi (University of Milan, ITALY)

Abstract. Let G be a finite group. An element g of G is called *vanishing element* if there exists an irreducible character χ of G such that $\chi(g) = 0$; in this case, we say that the conjugacy class of g is a vanishing conjugacy class. In this talk we discuss some arithmetical properties concerning the sizes of the vanishing conjugacy classes. This context is neatly portrayed by the *prime graph* of G for class sizes.

15:30–16:15 The powerful class of groups
Primož Moravec (University of Ljubljana, SLOVENIA)

Abstract. The notion of powerful class of finite p-groups was coined by Avinoam Mann in 2011. In this talk we focus on pro-p groups of finite powerful class. These groups are p-adic analytic, and the torsion elements always form a subgroup. It is shown that there are only finitely many finite p-groups of fixed coclass and powerful class.

We also sketch how the above results provide an upper bound for the p-length of a finite p-solvable group in terms of the powerful class of its Sylow p-subgroup.

17:00–17:45 Connectedness of the rank graph and properties of almost simple groups
Daniele Nemmi (University of Padua, ITALY)

Abstract. The generating graph of a group G is the graph on the elements of G in which x and y are joined if $G = \langle x, y\rangle$; it encodes how generating pairs are spread within the elements of the group. This graph has been studied by several authors and a lot is known about it. It has been conjectured for a long time that the graph induced by its non-isolated vertices is connected for every finite group. We will talk about this conjecture: we present a reduction to groups without abelian normal subgroups and the proof of a weaker version of this conjecture in the setting of the rank graph. This graph has vertex set the elements of the groups and x and y are joined if they belong to a generating set of minimal cardinality. When this cardinality (which we denote by $d(G)$) is 2, the rank graph coincides with the generating graph, so the former can be viewed as a generalisation of the latter to groups which are not necessarily 2-generated. In a paper in preparation and in [Chap. 4, 2], jointly with A. Lucchini, we prove the following.

Theorem 1. *The graph induced by the non-isolated vertices of the rank graph is connected for every finite group G such that $d(G) \geq 3$.*

Statements about connectedness of these kind of graphs are usually related to the structure of almost simple groups. We will present a fundamental ingredient of the proof of the theorem above, which is in fact a general result about almost simple groups, proved jointly with M. Costantini and A. Lucchini in [1].

Theorem 2. *Let G a finite almost simple group with socle G_0. If G/G_0 is abelian, then G contains an abelian subgroup A such that $G = AG_0$.*

July 24, 2024

11:30–12:15 Graphs defined on groups: some interactions
Peter Cameron (University of St Andrews, UK)

Abstract. My topic is not Cayley graphs, but graphs with vertex set the group, which capture some of the group structure. The oldest of these is the *commuting graph*, in which two elements are joined if they commute. Used by Brauer and Fowler in the seminal 1955 paper showing that there are only finitely many finite simple groups with a given involution centraliser, this was perhaps the first step to the Classification of the Finite Simple Groups. Other graphs defined since include the power graph (two elements joined if one is a power of the other), the enhanced power graph, nilpotency graph, and solubility graph (two elements joined if the group they generate is cyclic, nilpotent or soluble respectively), together with various contractions of these. Other graphs depend on generation properties: for example, in the generating graph, two elements are adjacent if they generate the group.

I will focus on several types of result linking these graphs to the groups:

- The graphs may be used to prove a group-theoretic result, as in the Brauer–Fowler theorem. I will give a new result of this type.
- Equality of two of the graphs often define interesting classes of groups, such as EPPO groups, Dedekind groups, and 2-Engel groups.
- Finally, it can happen that one of these graphs contains an interesting graph (such as one with large girth) buried within it.

12:30–12:50 On the diameter of graphs associated with groups
Michele Gaeta (University of Salerno, ITALY)

Abstract. The association of graphs to groups goes back to the 19th century when Cayley in [2] introduced a graph that encodes the abstract structure of a group. Later other graphs have been considered. More precisely, given a group property $\mathcal{P}$ and a group G one can consider the graph whose set of vertices is G and two vertices x and y are adjacent if and only if the subgroup generated by x and y has the property $\mathcal{P}$. In particular if the property $\mathcal{P}$ denotes solubility or commutativity, then the resulting graphs are called the solubility graph and the commutativity graph of the group G, respectively.

In discussing the connection and the diameter of such graphs, it is customary to exclude the unit of the group and sometimes all universal vertices. For instance, some results related to connectivity and diameter can be found in [1] and [3].

This talk aims to provide an overview of problems concerning the diameter of graphs associated with groups, with particular attention to the solubility graph and the commutativity graph.

14:30–15:15 On the Gowers trick for classical simple groups
Attila Maróti (Hun-Ren Alfréd Rényi Institute of Mathematics, HUNGARY)

Abstract. Let A, B, C be subsets of a finite group G. Let $\mathrm{Prob}(A,B,C)$ be the probability that if a and b are uniformly and randomly chosen elements from A and B respectively, then $ab \in C$. Let k be the minimal degree of a non-trivial complex irreducible character of G.

Gowers [1] proved the following beautiful theorem.

Theorem 3 (Gowers)**.** *If $\eta > 0$ is such that $|A||B||C| > |G|^3/\eta^2 k$, then*

$$(1-\eta)\frac{|C|}{|G|} < \mathrm{Prob}(A,B,C) < (1+\eta)\frac{|C|}{|G|}.$$

In this talk we will consider the special case when G is a classical simple group and when A, B, C are special kinds of sets. We will require at least two of these sets to be normal. (Recall that a subset of a group G is normal if it is invariant under conjugation by every element of G.) This is joint work [2] with Francesco Fumagalli.

15:30–16:15 Critical groups
Daniela Bubboloni (University of Florence, ITALY)

Abstract. The concept of critical class arises naturally in the reconstruction of the directed power graph of a finite group G by its undirected counterpart ([1], [2]). The presence in G of such classes makes that reconstruction for G a much more difficult task, with respect to groups having no critical class. That fact suggests that those $x \in G$ such that their equivalence class $[x]_{\mathtt{N}}$, with respect to the closed twin relation in the power graph, is critical could have important impact on properties and structure of G. We call such elements the critical elements of G and study their properties. We then consider the extreme case of groups $G \neq 1$ having all their nontrivial elements critical and call such groups *critical groups*. We characterize the critical groups, giving a complete description of their structure.

17:00–17:20 Vertex-transitive graphs and derangements

Marco Barbieri (University of Pavia, ITALY)

Abstract. A derangement is a permutation without any fixed points. An outstanding property of derangements in a finite transitive permutation group is their abundance. What can we say about the portion of derangements of a group acting transitively on the vertices of a graph? In the present talk, we will introduce a lower bound on the portion of derangements depending only on the valency of the graph, and we will discuss its asymptotic behaviour compared to other general lower bounds.

17:30–18:15 Representations of Finite Groups on Posets, Lattices, and Distributive Lattices

Pablo Spiga (University of Milano-Bicocca, ITALY)

Abstract. The objective of this presentation is to highlight recent advancements in refining results initially proposed by Babai regarding representations of finite groups as automorphism groups of posets, lattices, and distributive lattices. These refinements employ an indirect method, utilizing the concept of a Haar graph.

Let R be a group, and let S be a subset of R. The Haar graph $\mathrm{Haar}(R, S)$ of R with connection set S is defined as a graph with a vertex set of $R \times \{-1, 1\}$, where two distinct vertices $(x, -1)$ and $(y, 1)$ are deemed adjacent if and only if $yx^{-1} \in S$. The automorphism group of $\mathrm{Haar}(R, S)$ contains a subgroup isomorphic to R. When the automorphism group of $\mathrm{Haar}(R, S)$ equals R, the Haar graph $\mathrm{Haar}(R, S)$ is termed a Haar graphical representation of the group R.

During this seminar, we delve into the methodology employed to classify finite groups admitting Haar graphical representations. Subsequently, we demonstrate how this classification facilitates enhancements to a 1980 result by Babai regarding group representations on posets and distributive lattices, thereby achieving optimal outcomes in this domain.

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