Inverse Problems
Special Session B18

Luca Rondi
University of Pavia, ITALY

Jingni Xiao
Drexel University, USA

Inverse problems involve the task of deducing, from observed data, the priorly unknown causal factors that contributed to the data. They are ubiquitous in science and technology, with applications ranging from medical imaging to engineering, as in nondestructive evaluation, to geophysical prospections, just to name a few examples. Mathematically speaking, it entails determining information on a PDE by collecting measurements associated with its solutions. For the applications, it is crucial to have a deep theoretical understanding of inverse problems as well as reliable and efficient numerical reconstruction methods. Both these issues are mathematically very challenging, due to the fact that inverse problems are often nonlinear and ill-posed.

Our special session will focus on key aspects of inverse problems including uniqueness and stability issues as well as reconstruction methods. We will also explore the integration of machine learning tools in this field. We bring together a balanced mix of leading experts in the field and emerging researchers at the Ph.D. or postdoc level. To foster and strengthen the cooperation between the US and Italian inverse problems communities, both groups are well represented in our session.

For more information visit `mate.unipv.it/rondi/AMSUMI-Inverse_Problems.html`.

Schedule and Abstracts

July 25, 2024

11:30–12:15 The anisotropic fractional Calderón problem
Gunther Uhlmann (University of Washington, USA, and HKUST, Hong Kong, CHINA)

Abstract. We discuss some recent progress on the anisotropic Calderón problem for the fractional Laplacian.

12:30–12:50 Stability for two Coefficient Identification Problems
Sonia Foschiatti (University of Trieste, ITALY)

Abstract. In inverse problems, we are interested in analysing the stability of solutions in relation to observed data.

This talk reviews stability estimates for two coefficient identification problems in an anisotropic setting: the Calderón problem (or inverse conductivity problem) and an inverse boundary value problem modeled by the Schrödinger-type equation

$$\mathrm{div}(\sigma\nabla u) + qu = 0.$$

We present two Lipschitz stability estimates derived using the method of singular solutions and quantitative estimates of unique continuation. The first result is obtained in collaboration with Eva Sincich and Romina Gaburro.

References

[1] S. Foschiatti, R. Gaburro, E. Sincich *Stability for the Calderón's problem for a class of anisotropic conductivities via an ad hoc misfit functional*, Inverse Problems, 37 (2021), Paper No. 125007, 34 pp.

[2] S. Foschiatti, *Lipschitz stability estimate for the simultaneous recovery of two coefficients in the anisotropic Schrödinger type equation via local Cauchy data*, J. Math. Anal. Appl., 531 (2024), Paper No. 127753, 35 pp.

14:30–14:50 Nonlocality in inverse problems
Giovanni Covi (University of Helsinki, FINLAND)

Abstract. We will discuss some general aspects of inverse problems for nonlocal operators. In particular, we will consider the fundamental example of the fractional Calderòn problem, in which an electric potential has to be recovered from nonlocal Dirichlet-to-Neumann data. We will see how the nonlocality of the operator helps in the resolution of the problem, by allowing the use of a surprisingly powerful approximation technique. Finally, we will discuss some interesting applications, results and open problems.

15:00–15:20 Inverse Problems For Third-Order Nonlinear Perturbations Of Biharmonic Operators
Suman Sahoo (ETH Zurich, SWITZERLAND)

Abstract. In this talk, we discuss an inverse boundary problems for third-order nonlinear tensorial perturbations of biharmonic operators on a bounded domain in $\mathbb{R}^n$, where $n \geq 3$. By imposing appropriate assumptions on the nonlinearity, we demonstrate that the Dirichlet-to-Neumann map, known on the boundary of the domain, uniquely determines the genuinely nonlinear tensorial third-order perturbations of the biharmonic operator. The proof relies on the inversion of certain generalized momentum ray transforms on symmetric tensor fields. Notably, the corresponding inverse boundary problem for linear tensorial third-order perturbations of the biharmonic operator remains an open question. This is a joint work with Sombuddha Bhattacharyya, Katya Krupchyk, and Gunther Uhlmann.

15:30–16:15 The stability issue in inverse problems
Romina Gaburro (University of Limerick, IRELAND)

Abstract. We discuss the issue of stability in inverse problems. Given their ill-posed (and often non-linear) nature, it is necessary to reformulate the issue of stability, the continuous dependence of the unknown relevant physical parameter on the data, within the theory of ill-posed problems. This requires the need to impose *a-priori* information on the unknown parameter that is physically meaningful to the application in mind and that allows to restore stability in the inverse problem in question. As is well known the matter of stability is of fundamental importance in the reliability of any reconstruction procedure of the physical parameter since, in practice, the data/measurements of the problem will be affected by errors.

When the material occupying a domain under investigation $\Omega \subset \mathbb{R}^n$ is anisotropic (its physical property of interest depends on direction), there is a further fundamental obstruction to uniqueness in the inverse problem, due to the fact that any diffeomorphism of $\bar{\Omega}$ that keeps its boundary fixed, changes the physical property of interest in Ω but this changed is not visible through the boundary measurements.

In this talk we will address the above issues in inverse problems and give some positive answers to the questions of uniqueness and stability (and therefore reconstruction) in anisotropic inverse problems.

References

[1] G. Alessandrini, R. Gaburro, E. Sincich *Determining an anisotropic conductivity by boundary measurements: stability at the boundary*, J. Differential Equations, 382 (2024), 115–140.

[2] S. Foschiatti, R. Gaburro, E. Sincich *Stability for the Calderón's problem for a class of anisotropic conductivities via an ad hoc misfit functional*, Inverse Problems, 37(12) (2021), 125007.

17:00–17:20 Shape reconstruction for a planar conductivity inclusion
Doosung Choi (Louisiana State University, USA)

Abstract. This presentation focuses on the shape reconstruction of a conductivity inclusion. Our goal is to use generalized polarization tensors (GPTs), derived from external measurements, to mathematically reconstruct a homogeneous inclusion with a constant conductivity. The main achievement is developing a formula to represent conformal mapping coefficients by GPTs. To obtain this formula, we establish matrix factorizations for the GPTs.

Theorem 1 (Conformal mapping recovery)**.** *Let* Ω *be a simply connected planar domain with Lipschitz boundary.* Ω *is occupied with homogeneous material of conductivity* σ_c*, i.e.,* $\lambda = \frac{\sigma_c+\sigma_m}{2(\sigma_c-\sigma_m)}$*. The coefficients of the exterior conformal mapping associated with* Ω *satisfy*

$$\gamma^2 = \frac{\lambda}{2\pi}\left[\left(I-\overline{\mathbb{N}}_{1/2}\mathbb{N}_{1/2}\right)\left(I-4\lambda^2\overline{\mathbb{N}}_{1/2}\mathbb{N}_{1/2}\right)^{-1}\mathbb{N}^{(2)}\right]_{11},$$

$$a_0 = \frac{\left[\left(I-\overline{\mathbb{N}}_{1/2}\mathbb{N}_{1/2}\right)\left(I-4\lambda^2\overline{\mathbb{N}}_{1/2}\mathbb{N}_{1/2}\right)^{-1}\mathbb{N}^{(2)}\right]_{12}}{2\left[\left(I-\overline{\mathbb{N}}_{1/2}\mathbb{N}_{1/2}\right)\left(I-4\lambda^2\overline{\mathbb{N}}_{1/2}\mathbb{N}_{1/2}\right)^{-1}\mathbb{N}^{(2)}\right]_{11}},$$

$$a_m = \frac{\lambda^2}{\pi m}\sum_{n=1}^{m} p_{mn}\left[\mathbb{N}_{1/2}\left(I-\overline{\mathbb{N}}_{1/2}\mathbb{N}_{1/2}\right)\left(I-4\lambda^2\overline{\mathbb{N}}_{1/2}\mathbb{N}_{1/2}\right)^{-1}\mathbb{N}^{(2)}\right]_{n1}, \quad m \geq 1,$$

where $\mathbb{N}^{(1)}$, $\mathbb{N}^{(2)}$ *are GPTs and* $\mathbb{N}_{1/2} = \mathbb{N}^{(1)}\left(\mathbb{N}^{(2)}\right)^{-1}$*. Here,* $[\cdot]_{mn}$ *denotes an* (m,n)*-element of the given matrix.*

References

[1] D. Choi, J. Helsing, S. Kang, M. Lim *Inverse problem for a planar conductivity inclusion*, SIAM J. Imag. Sci., 16 (2), 969–995, 2023.

17:30–18:15 An inverse problem in monitoring of faults
Anna Mazzucato (Penn State University, USA)

Abstract. We discuss recovering the geometry and rock slippage of a buried seismic fault, modeled as an elastic dislocation, from surface displacement measurements. We discuss both the case in which the fault is purely linearly elastic and the case where the fault is assumed viscoelastic. This is also joint work with PhD student Arum Lee.

References

[1] A. Aspri, E. Beretta, M. de Hoop, and A. L. Mazzucato. Detection of dislocations in a 2D anisotropic elastic medium. *Rend. Mat. Appl. (7)*, 42(3-4):183–195, 2021.

[2] A. Aspri, E. Beretta, and A. L. Mazzucato. Dislocations in a layered elastic medium with applications to fault detection. *J. Eur. Math. Soc. (JEMS)*, 25(3):1091–1112, 2023.

[3] A. Aspri, E. Beretta, A. L. Mazzucato, and M. V. De Hoop. Analysis of a model of elastic dislocations in geophysics. *Arch. Ration. Mech. Anal.*, 236(1):71–111, 2020.

July 26, 2024

11:30–11:50 Learned regularization by denoising for Limited-Angle Computed Tomography
Andrea Sebastiani (University of Modena and Reggio Emilia, ITALY)

Abstract. Recent advancements in unfolded iterative methods allow for learning the parameters of the optimization algorithm along with a suitable pseudodifferential correction for aspects that cannot be addressed by model-based methods. This supervised learning modality can be extended to the proximal operator, leveraging the modularity of proximal splitting methods and replacing it with a denoiser. More specifically, by considering a particular class of denoisers, defined as a gradient step on a potential function, it is possible to derive a non-convex regularization term, whose proximal operator corresponds to the considered denoiser.
This characterization enables the convergence analysis of the resulting Plug-and-Play (PnP) scheme and the study of potentially accelerated variants, to reduce the number of iterations required to obtain a good solution. The numerical experiments on limited-angle computed tomography (CT) show promising results, demonstrating the benefit of embedding of sophisticated and complex image priors through expressive denoisers.
Joint work with Tatiana A Bubba, Luca Ratti, and Subhadip Mukherjee.

References

[1] Kamilov, U. S., Bouman, C. A., Buzzard, G. T., Wohlberg, B, *Plug-and-play methods for integrating physical and learned models in computational imaging: Theory, algorithms, and applications*, IEEE Signal Processing Magazine, 40.1 (2023), 85-97.

[2] Hurault, S., Leclaire, A., Papadakis, N., *Proximal denoiser for convergent plug-and-play optimization with nonconvex regularization*, International Conference on Machine Learning (2022), 9483-9505.

[3] Bubba, T. A., Galinier, M., Lassas, M., Prato, M., Ratti, L., Siltanen, S., *Deep neural networks for inverse problems with pseudodifferential operators: An application to limited-angle tomography*, SIAM Journal on Imaging Sciences, 14.2 (2021), 470-505.

12:00–12:45 Compressed sensing for the sparse Radon transform
Matteo Santacesaria (University of Genoa, ITALY)

Abstract. Compressed sensing allows for the recovery of sparse signals from few measurements, whose number is proportional, up to logarithmic factors, to the sparsity of the unknown signal. The classical theory mostly considers either random linear measurements or subsampled isometries. In particular, the case with the subsampled Fourier transform finds applications to undersampled magnetic resonance imaging. In this talk, I will show how the theory of compressed sensing can also be rigorously applied to the sparse Radon transform, in which only a finite number of angles are considered. One of the main novelties consists in the fact that the Radon transform is associated to an ill-posed inverse problem, and the result follows from a new theory of compressed sensing for abstract inverse problems. This is a joint work with G.S. Alberti, A. Felisi and S.I. Trapasso.

References

[1] G.S. Alberti, A. Felisi, M. Santacesaria, S.I. Trapasso, *Compressed sensing for inverse problems and the sample complexity of the sparse Radon transform*, preprint arXiv:2302.03577, 2023.

14:30–15:15 Inverse magnetisation problem in paleomagnetic context
Dmitry Ponomarev (Centre Inria d'Université Côte d'Azur, FRANCE)

Abstract. The process of extraction of relict magnetic information from geosamples and meteorites is a challenging but important task in paleomagnetic research. Due to the weak intensity of the field produced by a magnetised rock, the measurements have to be performed in direct vicinity of the sample and using highly sensitive magnetometric devices such as SQUID and QDM. The basic quantity of interest is the net magnetisation (magnetisation moment vector). Reconstruction of this quantity hinges on effective processing of the experimental data, with the main challenges being the limited measurement area and the noise contamination. Motivated by a concrete experimental setting in the Paleomagnetism lab at EAPS department of MIT (USA), we will focus on constructive issues. Namely, using asymptotic analysis, one can obtain explicit formulas estimating the net magnetisation vector. However, since the measurement area is usually not sufficiently large, we face an intermediate problem of the field extrapolation. We propose and analyse some extrapolation strategies and illustrate them numerically.

References

[1] L. Baratchart, D.P. Hardin, E.A. Lima, E.B. Saff, B.P. Weiss, Characterizing kernels of operators related to thin-plate magnetizations via generalizations of Hodge decompositions, *Inverse Problems* **29** (2013).

[2] D. Ponomarev, Magnetisation moment of a bounded 3D sample: asymptotic recovery from planar measurements on a large disk using Fourier analysis, arXiv:2205.14776, 2022.

15:30–16:15 Mathematical analysis of an inverse problem arising in a model of prostate cancer growth
Elena Beretta (New York University Abu Dhabi, UAE)

Abstract. The availability of cancer measurements over time enables the personalised assessment of tumour growth and therapeutic response dynamics. However, many tumours are treated after diagnosis without collecting longitudinal data, and cancer monitoring protocols may include infrequent measurements. To facilitate the estimation of disease dynamics and better guide ensuing clinical decisions, we investigate an inverse problem enabling the reconstruction of earlier tumour states by using a single spatial tumour dataset and a biomathematical model describing disease dynamics. We focus on prostate cancer, since aggressive cases of this disease are usually treated immediately after diagnosis. We describe the tumour evolution with a phase-field model driven by a generic nutrient ruled by reaction-diffusion dynamics. The model is completed with another reaction-diffusion equation for the local production of prostate-specific antigen, which is a key prostate cancer biomarker. We first improve previous well-posedness results by further showing that the solution operator is continuously Fréchet differentiable. We then analyse the backward inverse problem concerning the reconstruction of earlier tumour states starting from measurements of the model variables at the final time. Since this problem is severely ill-posed, only very weak conditional stability of logarithmic type can be recovered from the terminal data. However, by restricting the unknowns to a compact subset of a finite-dimensional subspace, we can derive an optimal quantitative Lipschitz stability estimate.

References

[1] E. Beretta, C. Cavaterra, M. Fornoni, G. Lorenzo, E. Rocca, *Mathematical analysis of a model-constrained inverse problem for the reconstruction of early states of prostate cancer growth*, April 2024, https://arxiv.org/abs/2404.12198

17:00–17:20 Full discretization and regularization for the Calderón problem

Alessandro Felisi (University of Genoa, ITALY)

Abstract. In this talk, we consider the Calderón problem with discontinuous conductivities. We present a fully discretized variational approach that involves a data fidelity term and a total variation penalty term with a corresponding regularization parameter. The discretization encompasses the boundary measurements, using the complete electrode model, the unknown conductivity and the potential. The two key parameters in the analysis are related to the electrodes size and the mesh resolution. Our analysis establishes a precise method for selecting the discretization and the regularization parameters based on the noise level to ensure that the solution to the discretized problem remains meaningful. Notably, we find that both electrode and mesh size parameters should exhibit a polynomial decay with respect to the noise level. This is joint work with Luca Rondi (Pavia).

References

[1] A. Felisi, L. Rondi, *Full discretization and regularization for the Calderón problem*, arxiv preprint, arXiv:2112.11489, 2021.

17:30–18:15 Reduced order models and the Lippmann Schwinger Lanczos method in inverse scattering

Shari Moskow (Drexel University, USA)

Abstract. We combine data-driven reduced order models with the Lippmann-Schwinger integral equation to produce a direct nonlinear inversion method. The ROM is viewed as a Galerkin projection and is sparse due to Lanczos orthogonalization. Embedding into the continuous problem, we produce an approximation of the internal solution directly from the data. This internal solution is then used in the Lippmann-Schwinger equation. The approach allows us to process more general transfer functions than the earlier versions of the ROM based inversion algorithms. We give examples of its use for spectral domain MIMO problems and in the time domain given mono static data, targeting synthetic aperture radar. For radar problems, we also show a new technique of data completion of monostatic data to full MIMO, to further improve the internal solution.
Joint work with Vladimir Druskin and Mikhail Zaslavsky.

E–mail: luca.rondi@unipv.it.