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Factorization Algebras and Geometry
Special Session A5

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The space of conformal blocks is an object that describes the collection of all correlation function in a two-dimensional conformal field theory. Mathematically, it is a vector space computed from the input of a pointed curve X (the "space-time"), a vertex algebra V (which encodes the chiral conformal field theory), a collection V-representations, and possibly a principal G-bundle P on X (the gauge bundle), where G is an algebraic group. As the curve X varies in $\mathcal{M}_{g,n}$, and the bundle P in $\mathrm{Bun}_{G,X}$—the moduli space of principal G-bundles on X—spaces of conformal blocks give rise to interesting sheaves equipped with projectively flat connection (the Knizhnik–Zamolodchikov connection).

A particularly beautiful instance of this construction happens in the case of the Wess–Zumino–Witten model of conformal field theory. Here, the representation-theoretic input (the vertex algebra and its modules) correspond to an affine Kac-Moody algebra at positive integer level, and the space of conformal blocks may be identified with spaces of non-abelian theta functions on $\mathrm{Bun}_{G,X}$ and its parabolic cousins. A rich combinatorial structure emerges which allows for the computation of the dimensions of spaces of conformal blocks through the celebrated Verlinde formula.

In the last 2-3 decades, new rigorous geometric approaches to conformal field theory have emerged, allowing the conformal block construction to be re-cast and generalized in several directions. First, in the theory of chiral algebras, developed by Beilinson and Drinfeld, conformal blocks appear as the *zeroth chiral homology*—one of a (potentially infinite) tower of cohomology groups. It is an interesting problem, both purely mathematical and in physical contexts, to interpret these 'higher' conformal blocks. Similarly, in the formalism of factorization algebras developed by Costello and Gwilliam, they appear as the *factorization homology*. This framework also allows one to treat higher-dimensional examples through the use of derived geometry, and applies to more general classes of quantum field theories.

The aim of this special session is to bring together experts in the theory of these generalized conformal blocks with a view towards their geometric properties, and to explore how the rich interplay of representation theory, moduli spaces, and combinatorics, may be productively generalized in these settings.

Schedule and Abstracts

July 23, 2024

11:00–11:45 Topological modular forms and conformal field theories
Du Pei (Centre for Quantum Mathematics, University of Southern Denmark, Denmark)
Abstract. We survey some applications of the theory of topological modular forms to the study of quantum fields theories, especially conformal and topological ones, in two and three dimensions.

12:00–12:45 The modular functor perspective on spaces of conformal blocks
Lukas Woike (Université de Bourgogne, France)
Abstract. *Modular functors* are consistent systems of projective mapping class group representations, see e.g. [2] for an introduction. In particular, to each compact oriented surface equipped with labels for its boundary components, one associates a vector space with a projective representation of the mapping class group of that surface. This vector space is called the *space of conformal blocks* for that surface. In my talk, I will describe modular functors using modular operads. Moreover, I will explain how factorization homology can be used to classify modular functors. The talk is on joint work with Adrien Brochier [3], and is partially also based on joint work with Lukas Müller [4] [5].

In more detail, we will identify genus zero modular functors with values in a symmetric monoidal bicategory $\mathcal{S}$ with cyclic $\mathcal{S}$-valued algebras over the framed little disks operad fE_2. These are characterized in [4], and they always extend uniquely to an *ansular functor*, i.e. a modular algebra over the modular operad of handlebodies [5].

One has the following weak uniqueness result for extensions from genus zero modular functors to modular functors defined at all genera:

Theorem 1 ([3]). *The space of extensions of a genus zero modular functor to a modular functor is contractible if it is non-empty.*

This is a far-reaching generalization of [1].

Those cyclic fE_2-algebras that turn out to extend to higher genus are exactly the ones for which the so-called *skein modules*, that in this context will be defined using factorization homology, associated to the different handlebodies with the same boundary surface are isomorphic. We call these cyclic fE_2-algebras *connected*. We then arrive at:

Theorem 2 ([3]). *The moduli space of $\mathcal{S}$-valued modular functors is equivalent to the 2-groupoid of connected cyclic $\mathcal{S}$-valued fE_2-algebras.*

References

[1] J. E. Andersen, K. Ueno. Modular functors are determined by their genus zero data. *Quantum Topol.* 3(4):255–291, 2012.

[2] B. Bakalov, A. Kirillov. Lectures on tensor categories and modular functors. University Lecture Series. Volume 21. Am. Math. Soc. 2001.

[3] A. Brochier, L. Woike. A Classification of Modular Functors via Factorization Homology. arXiv:2212.11259 [math.QA]

[4] L. Müller, L. Woike. Cyclic framed little disks algebras, Grothendieck-Verdier duality and handlebody group representations. *Quart. J. Math.* 74(1):163–245, 2023.

[5] L. Müller, L. Woike. Classification of Consistent Systems of Handlebody Group Representations. *Int. Math. Res. Not.* 2024(6):4767–4803, 2023.

14:30–15:15 Higher rank series invariants for plumbed 3-manifolds
Nicola Tarasca (Virginia Commonwealth University, USA)
Abstract. The Witten-Reshetikhin-Turaev (WRT) invariants provide a powerful framework for constructing a family of invariants for framed links and 3-manifolds starting from the data of a modular tensor category and surgery presentations for the 3-manifolds. An ongoing pursuit in quantum topology revolves around the categorification of these invariants. Recent progress has

been made in this direction, particularly through a physical definition of a new series invariant for a class of plumbed 3-manifolds. These invariants exhibit a convergence towards the WRT invariants in their limits. In this talk, I will present a refinement of such series invariants and show how one can obtain infinitely many new series invariants starting from the data of a root lattice of rank at least 2. This is joint work with Allison Moore.

15:30–16:15 Vertex algebras from divisors on Calabi-Yau threefolds
Dylan Butson (University of Oxford, UK)

*Abstract.*I will explain two conjecturally equivalent constructions of vertex algebras associated to divisors S on certain toric Calabi-Yau threefolds Y, and some partial results towards the proof of their equivalence. One construction is geometric, as a convolution algebra acting on the homology of certain moduli spaces of coherent sheaves supported on the divisor, following the proof of the AGT conjecture by Schiffmann-Vasserot and its generalization to divisors in C^3 by Rapcak-Soibelman-Yang-Zhao. The other is algebraic, as the kernel of screening operators on lattice vertex algebras determined by the GKM graph of Y and a Jordan-Holder filtration of the structure sheaf of S. This provides a correspondence between the enumerative geometry of coherent sheaves on Calabi-Yau threefolds and the representation theory of W-algebras and affine Yangian-type quantum groups.

17:00–17:45 Towards a Dolbeault AGT correspondence
Surya Raghavendran (Yale University, USA)

Abstract. In seminal work, Grojnowski-Nakajima constructed an action of the Heisenberg algebra on equivariant cohomology of Hilbert schemes. I will describe two holomorphic factorization algebras in three complex dimensions that furnish higher dimensional uplifts of the Heisenberg and Virasoro vertex algebras respectively. Conjecturally, mode algebras of these factorization algebras act on coherent cohomology of moduli of twisted Higgs sheaves on surfaces, and in a particular example, the action admits a cohomological deformation to the one studied by Grojnowski-Nakajima. I will describe motivation and evidence for this conjecture, rooted in a new mathematical understanding of a nebulous superconformal field theory in six dimensions.

July 24, 2024

11:30–12:15 Raviolo vertex algebras
Niklas Garner (University of Washington, Seattle, USA)

Abstract. I will describe recent work with B. Williams about an algebraic structure modeling the local observables in mixed holomorphic-topological quantum field theories in three dimensions. The resulting algebraic structure is directly analogous to a vertex algebra, but where holomorphic functions on a punctured complex curve are replaced by (derived) functions on a punctured 3-manifold that are constant along the leaves of chosen transverse holomorphic foliation. Our construction provides a geometric interpretation of a vertex-algebraic structure shown by Oh and Yagi to describe local operators in holomorphic-topological quantum field theories.

Theorem 3 (NG, BW)**.** *Let V be a vector space. The structure of a raviolo vertex algebra on V is equivalent to that of a 1-shifted Poisson vertex algebra.*

Due to their formal similarities, many vertex-algebraic constructions pass over to the setting of raviolo vertex algebras. For example, there are raviolo analogues of free-field algebras, affine Kac-Moody algebras, and Virasoro and superconformal algebras. Time permitting, I will describe one such construction appearing in recent work with S. Raghavendran and B. Williams whereby we realize.

Theorem 4 (NG, SR, BW)**.** *To any $\mathcal{N} = 2$ superconformal raviolo vertex algebra V there are two canonically defined 2-shifted Poisson schemes $\mathcal{M}_H[V]$ and $\mathcal{M}_C[V]$ called its Higgs and Coulomb branches.*

References

[1] N. Garner, B. Williams *Raviolo vertex algebras*, (2023).

[2] J. Oh, J. Yagi, *Poisson vertex algebras in supersymmetric field theories*, Lett. Math. Phys., 110 (2020), 2245–2275.

[3] N. Garner, S. Raghavendran, B. Williams *Higgs and Coulomb branches from superconformal raviolo vertex algebras*, (2023).

12:20–13:05 Raviolo vertex algebras and conformal blocks
Charles Young (University of Hertfordshire, UK)

Abstract. One very promising approach to higher conformal blocks is through factorization algebras in their various incarnations, smooth and algebraic.

In this talk, though, I'll discuss an attempt to do something more naive and direct: namely to generalize the usual definition of rational conformal blocks/coinvariants on the Riemann sphere by simply replacing the various algebras that appear – for example, the algebra of regular functions on the configuration space of marked points – by their derived analogs in higher settings.

The case of raviolo vertex algebras – in the sense recently introduced by Garner and Williams – is an attractive arena in which to try this, because it is a comparatively mild generalization of the usual situation in complex dimension one.

I'll introduce a notion of configuration space in the raviolo setting, and construct a model in dg algebras of the derived global sections of its structure sheaf, by using the Thom-Sullivan functor. Using that, I'll define a space of coinvariants in the raviolo setting, and finally show that the state-field map of raviolo vertex algebras emerges in the limit in which marked points collide.

This talk is based in part on joint work `https://arxiv.org/abs/2401.11917` with Luigi Alfonsi and Hyungrok Kim.

14:30–15:15 First chiral homology on $\mathbb{P}^1$ and V-module extensions
Juan Guzman (IMPA, Rio de Janeiro, Brazil)

Abstract. In this talk I will present the construction of an isomorphism between equivalence classes of extensions $0 \to A \to B \to C \to 0$ of V-modules, where V is a vertex operator algebra, and the dual space of the first chiral homology group of V over $\mathbb{P}^1$ with coefficients $A^\vee$ and C inserted at ∞ and 0 respectively.

This work is part of an ongoing project on chiral homology at IMPA, together with Reimundo Heluani and Thadeu Henrique Cardoso.

15:30–16:15 Hodge filtrations in chiral homology
Jethro van Ekeren (IMPA, Rio de Janeiro, Brazil)

Abstract. Many of the deepest results in the representation theory of vertex algebras revolve around the structure of spaces of conformal blocks (i.e., of chiral homology in degree zero) at the boundary of the moduli space of elliptic curves. In this talk I will present some results in this direction on chiral homology in higher degree, deduced from analysis of a sort of Hodge filtration on the associated chiral chain complex. This is joint work with R. Heluani.

17:00–17:45 Factorization algebras in the geometric Langlands correspondence
Nick Rozenblyum (University of Toronto, Canada)

Abstract. The geometric Langlands correspondence is an equivalence of appropriate categories of sheaves on moduli spaces associated to a smooth proper algebraic curve and a reductive group G. The theory of factorization algebras gives a "local-to-global" approach to studying these categories.I will give an overview of the proof of the geometric Langlands correspondence focusing on the local-to-global aspects which are of independent interest. This is joint work with Arinkin, Beraldo, Chen, Færgeman, Gaitsgory, Lin, and Raskin.

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