Functional inequalities, shape optimization and elliptic PDEs
Special Session B13

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Optimal functional inequalities often encode important information about the underlying ambient space, which could be critical for both abstract and applied research. Sharp Sobolev type inequalities, for instance, are fundamental tools in analysis, geometry, and mathematical physics. They are used to determine the existence and qualitative behavior of solutions to certain nonlinear PDEs, they are deeply related to isoperimetric problems and many optimal eigenvalue bounds, given in terms of the geometry of the background manifold. The purpose of this special session is to bring together specialists working on geometric and functional inequalities, shape optimization, and related PDEs, and to encourage lively discussions leading to further developments and open problems.

Schedule and Abstracts

July 25, 2024

11:30–11:50 *Stability for the logarithmic Hardy-Littlewood-Sobolev Inequality with application to the Keller-Segel system*

Eric Carlen, Rutgers University, USA

Abstract. We prove an optimal stability bound for the Onofri inequality, and then apply a duality method to prove an optimal stability theorem for the logarithmic Hardy-Littlewood-Sobolev inequality. We then apply this to the estimation of the rate of approach to equilibrium for the critical mass Keller-Segel system.

12:00–12:20 *Some New Inequalities in Analysis and Geometry*

Changfeng Gui, University of Tennessee, USA

Abstract. The classical Moser-Trudinger inequality is a borderline case of Sobolev inequalities and plays an important role in geometric analysis and PDEs in general. Aubin in 1979 showed that the best constant in the Moser-Trudinger inequality can be improved by reducing to one half if the functions are restricted to the complement of a three dimensional subspace of the Sobolev space H^1, while Onofri in 1982 discovered an elegant optimal form of Moser-Trudinger inequality on sphere. In this talk, I will present new sharp inequalities which are variants of Aubin and Onofri inequalities on the sphere with or without mass center constraints.

One such inequality, for example, incorporates the mass center deviation (from the origin) into the optimal inequality of Aubin on the sphere, which is for functions with mass centered at the origin. The main ingredient leading to the above inequalities is a novel geometric inequality: Sphere Covering Inequality.

Efforts have also been made to show similar inequalities in higher dimensions. Among the preliminary results, we have improved Beckner's inequality for axially symmetric functions when the dimension $n = 4, 6, 8$. Many questions remain open.

The talk is based on collaboration with Amir Moradifam, Sun-Yung Alice Chang, Yeyao Hu, Weihong Xie, Tuoxin Li, Juncheng Wei, And Zikai Ye.

12:30–12:50 *Symmetrization results for general nonlocal linear ellipitic and parabolic problems*

Vincenzo Ferone, Università degli Studi di Napoli, Italy

Abstract. We discuss a Talenti-type symmetrization result in the form of mass concentration (*i.e.* integral comparison) for very general linear nonlocal elliptic problems, equipped with homogeneous Dirichlet boundary conditions. In this framework, the relevant concentration comparison for the classical fractional Laplacian can be reviewed as a special case of our main result, thus generalizing previous results obtained in collaboration with B. Volzone. Also a Cauchy-Dirichlet nonlocal linear parabolic problem is considered. The results are contained in a joint paper with G. Piscitelli and B. Volzone.

14:30–14:50 *Hausdorff measures and Orlicz-Sobolev maps*

Andrea Cianchi, Università degli Studi di Firenze, Italy

Abstract. A comprehensive theory of the effect of Orlicz-Sobolev maps, between Euclidean spaces, on subsets with zero or finite Hausdorff measure is offered. Arbitrary Orlicz-Sobolev spaces embedded into the space of continuous function and Hausdorff measures built upon general gauge functions are included in our discussion. An explicit formula for the distortion of the relevant gauge function under the action of these maps is exhibited in terms of the Young function defining the Orlicz-Sobolev space. New phenomena and features, related to the flexibility in the definition of the degree of integrability of weak derivatives of maps and in the notion of measure of sets, are detected. Classical results, dealing with standard Sobolev spaces and Hausdorff measures, are recovered, and their optimality is shown to hold in a refined stronger sense. Special instances available in the literature, concerning Young functions and gauge functions of non-power type, are also reproduced and, when not sharp, improved. This is the content of joint work with M.V.Korobkov and J.Kristensen.

15:00–15:20 *Embeddings for Fractional Orlicz-Sobolev spaces*

Angela Alberico, Consiglio Nazionale delle Ricerche, Napoli, Italy

Abstract. The optimal target space is exhibited for embeddings of fractional-order Orlicz-Sobolev spaces. Both the subcritical and the supercritical regimes are considered. In the former case, the smallest possible Orlicz target space is detected. In the latter, the relevant Orlicz-Sobolev spaces are shown to be embedded into the space of bounded continuous functions in $\mathbb{R}^n$. Moreover, their optimal modulus of continuity is exhibited.
These results are the subject of a series of joint papers with Andrea Cianchi, Luboš Pick and Lenka Slavíková.

15:30–15:50 *Blow-up and global solutions for a parabolic problem with Trudinger-Moser nonlinearity*

Federica Sani, Università degli Studi di Modena e Reggio Emilia

Abstract. We consider the Cauchy problem for a 2-space dimensional heat equation with exponential nonlinearity. More precisely, we consider initial data in $H^1(\mathbb{R}^2)$, and a square-exponential nonlinearity, which is critical in the energy space $H^1(\mathbb{R}^2)$ in view of the Trudinger-Moser inequality. By means of energy methods, we discuss the dichotomy between blow-up and global existence for solutions below the ground state energy level. The splitting between blow-up and global existence for low energies is determined by the sign of a suitable functional, and it is related to the corresponding Trudinger-Moser inequality. This is a joint work with Michinori Ishiwata (Osaka University), Bernhard Ruf (Istituto Lombardo Accademia di Scienze e Lettere), and Elide Terraneo (Università degli Studi di Milano).

16:00–16:20 *On the Maximum Principle for higher order operators*

Daniele Cassani, Università degli Studi dell'Insubria, Italy

Abstract. We discuss a general principle of perturbing higher order operators with lower order derivatives in order to restore the maximum principle in the framework in which it is well known to fail. This is somehow delicate and the main ingredient is a new Harnack-type inequality. We first prove De Giorgi type level estimates for functions in $W^{1,t}$, with $t > 2$. This augmented

integrability enables us to establish a new Harnack type inequality for functions which do not necessarily belong to De Giorgi's classes as obtained by Di Benedetto-Trudinger for functions in $W^{1,2}$. As a consequence, we prove the validity of the strong maximum principle for uniformly elliptic operators of any even order, in fairly general domains and in any dimension, provided either lower order derivatives or inertial effects are taken into account.

17:00–17:20 *From the isoperimetric inequality to functional inequalities under nonnegative curvature*

Gioacchino Antonelli, New York University, USA

Abstract. In this talk I will discuss recent results obtained with several collaborators on isoperimetric and functional inequalities on smooth (or nonsmooth) spaces with nonnegative curvature.

In particular, at first I will discuss a new proof of the sharp and rigid isoperimetric inequality on spaces with nonnegative curvature and maximal volume growth. Then, I will show how this inequality implies sharp and rigid functional inequalities.

17:30–17:50 *Moser-Trudinger inequalities on complete manifolds with large volume growth*

Liuyu Qin, Hunan University of Finance and Economics, China

Abstract. In this talk, I will discuss two Moser-Trudinger inequalities on complete Riemannian manifolds with nonnegative Ricci curvature and large volume growth. These inequalities will feature different best constants under different norm conditions. This is joint work with Luigi Fontana and Carlo Morpurgo.

July 26, 2024

11:30–11:50 *A sharp criterion for zero modes of the Dirac equation*

Micheal Loss, Georgia Institute of Technology, USA

Abstract. A sharp criterion for solutions of the Dirac equation $\gamma\cdot(-i\nabla - A)\psi = 0$ in d dimensions is discussed. It turns out that that $\|A\|^2_{L^d} \geq \frac{d}{d-2}\, S_d$ is a necessary condition for the existence of a nontrivial solution of the Dirac equation, where S_d is the sharp Sobolev constant. If d is odd and $\|A\|^2_{L^d} = \frac{d}{d-2}\, S_d$, then there exist vector potentials that allow for zero modes. A complete classification of these vector potentials and their corresponding zero modes can be given.

12:00–12:20 *New Inequalities for the Low Eigenvalues of the Vibrating Clamped Plate and Buckling Problems with Perturbations*

Mark Ashbaugh, University of Missouri - Columbia, USA

Abstract. We discuss new inequalities for the low eigenvalues of the vibrating clamped plate and the buckling of a clamped plate and perturbations of these. Both problems are eigenvalue problems for 4th order partial differential operators where the highest order part of the operator is the biharmonic operator, and in both cases we deal with the case of clamped boundary conditions.

12:30–12:50 *On a Bliss-Moser type inequality*

Bernhard Ruf, Istituto Lombardo-Accademia di Scienze e Lettere, Milano, Italy

Abstract. We derive a limiting inequality for the integral inequalities by Bliss. We then consider a critical version of this inequality which is of Moser type, and discuss related non-compactness properties. Furthermore, we show that this inequality is related to critical boundary growth for functions on a disk in two dimensions.

14:30–14:50 *Non-degeneracy of solutions for singular Liouville equations in dimension one*

Gabriele Mancini, Università degli Studi di Bari, Italy

Abstract. In this talk, I will discuss existence, classification and non-degeneracy results for solutions to singular Liouville-type equations of the form

$$(-\Delta)^{\frac{n}{2}} u = |x|^{n(\alpha-1)} e^{nu} \quad \text{in } \mathbb{R}^n. \tag{1}$$

In dimension one, the problem has applications in mathematical modelling of galvanic corrosion phenomena for ideal electrochemical cells consisting of an electrolyte solution confined in a bounded domain with an electrochemically active portion of boundary. In higher dimension, Lioville equations have applications to prescribed curvature problems in conformal geometry: solutions correspond to constant Q-curvature metrics on the Euclidean space, with a singular point at the origin.

After a general overview of the existing literature, I will focus on the one-dimensional case and I will prove that solutions of (1) are non-degenerate for $\alpha \in (0,1) \cup (1,2)$. Namely, the space of solutions in $H^{\frac{1}{2}}(\mathbb{R})$ to the linearized equation

$$(-\Delta)^{\frac{1}{2}}\varphi = |x|^{\alpha-1}e^u\varphi \quad \text{in } \mathbb{R} \tag{2}$$

has dimension one. The proof relies on the use of harmonic extensions and conformal transformations to rewrite the linearized equation (2) as a Steklov eigenvalue problem on either an intersection or a union of two disks, depending on the values of α.

These results are contained in joint works in collaboration with A. DelaTorre, A. Pistoia, A. Hyder and L. Martinazzi.

15:00–15:20 *Uniqueness of least-energy solutions to the fractional Lane-Emden equation in the ball*

Enea Parini, Aix-Marseille Université, France

Abstract. We prove uniqueness of least-energy solutions to the fractional Lane-Emden equation, under homogeneous Dirichlet exterior conditions, when the underlying domain is a ball $B \subset \mathbb{R}^N$. The equation reads as follows:

$$\begin{cases} (-\Delta)^s u &= u^p & \text{in } B, \\ u &= 0 & \text{in } \mathbb{R}^N \setminus B. \end{cases}$$

Here $s \in (0,1)$, and $p \in \left(1, \frac{N+2s}{N-2s}\right)$, which makes the nonlinearity superlinear and subcritical. The proof makes use of Morse theory and is inspired by some results obtained by C. S. Lin in the '90s. A new Hopf's Lemma-type result is an essential ingredient in order to prove nondegeneracy of least-energy solutions.

15:30–15:50 *Moser-Trudinger inequalities: from local to global*

Luigi Fontana, Università degli Studi di Milano-Bicocca, Italy

Abstract. Given a complete Riemannian manifold, we define the notion of local Moser-Trudinger inequality. We show that by imposing a stronger norm condition or by assuming the validity of the Poincaré inequality, the local Moser-Trudinger inequality implies the global one. Hadamard manifolds provide a significant application. Joint work with Carlo Morpurgo and Liuyu Qin.

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