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Recent Developments on Certain Evolution Partial Differential Equations
Special Session A22

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The goal of this section is to bring together researchers whose interest is in the study of a variety of evolution equations that represent fundamental physical phenomena. The study of these equations involves harmonic and Fourier analysis, geometry and stochastic calculus. We expect our speakers to give colloquium style talks involving the analysis of the Kolmogorov's operator, of many body systems of particles and wave packets, of invariant measures, and of singularity formations. Many of the problems discussed could be collected under the wide umbrella of kinetic theory and wave turbulence theory, two subjects that in recent years have seen an incredible number of fundamental advances.

For more information visit
`https://www.mathematical-analysis.unimore.it/jm-umi-ams/#SSA22`

Schedule and Abstracts

July 23, 2024

11:00–11:45 On the effective dynamics of Bose-Fermi mixtures
Nataša Pavlović (University of Texas Austin, USA)

Abstract. Investigating degenerate mixtures of bosons and fermions is an extremely active area of research in experimental physics for constructing and understanding novel quantum bound states such as those in superconductors, superfluids, and supersolids. These ultra-cold Bose-Fermi mixtures are fundamentally different from degenerate gases with only bosons or fermions. They not only show an enriched phase diagram, but also a fundamental instability due to energetic considerations coming from the Pauli exclusion principle. Inspired by this activity in the physics community, recently we started exploring the mathematical theory of Bose-Fermi mixtures. One of the main challenges is understanding the physical scales of the system that allow for suitable analysis. We will describe how we overcame this challenge by identifying a novel scaling regime in which the fermion distribution behaves semi-clasically, but the boson field remains quantum-mechanical. In this regime, the bosons are much lighter and more numerous than the fermions.

This is a Joint work in collaboration with Esteban Cárdenas (University of Texas Austin, USA) and Joseph Miller (University of Texas Austin, USA).

References

[1] E. Cárdenas, J.K. Miller, N. Pavlović, *On the effective dynamics of Bose-Fermi mixtures*, arXiv:2309.04638, Submitted for publication.

12:00–12:20 Quantum diffusion and approximate semigroups
Felipe Hernández (Massachusetts Institute of Technology, USA)

Abstract. The random Schrödinger equation is a toy model for waves in random media and for the motion of an electron in an imperfect metal. The evolution of smooth observables for this equation are described by a linear Boltzmann equation, which gives a classical picture for the motion of electrons in terms of a Markov process with discontinuous jumps in the momentum.

In this talk I will introduce both of these equations and discuss some of the ideas needed to rigorously the classical description from the quantum mechanical system.

12:30–12:50 Harnack inequalities for kinetic integral equations
Francesca Anceschi (Università Politecnica delle Marche, ITALY)

Abstract.
I will present some recent results for weak solutions to the kinetic integral equation

$$\big[\partial_t + v \cdot \nabla_x\big] f = \mathcal{L}_v f,$$

where the diffusion term $\mathcal{L}_v$ is an integro-differential operator, whose nonnegative kernel is of fractional order $s \in (0, 1)$ and has measurable coefficients. In particular, I will focus on nonlocal Harnack inequalities for nonnegative weak solutions f not requiring the usual a priori boundedness assumption. The talk is based on a series of papers by Kassmann, Weidner, Palatucci, Piccinini, and myself.

13:00–14:30 **Lunch Break**

14:30–14:15 Self-similar blow up profiles for fluids via physics-informed neural networks
Javier Gómez-Serrano (Brown University, Providence, USA)

Abstract. In this talk I will explain a new numerical framework, employing physics-informed neural networks, to find a smooth self-similar solution (or asymptotically self-similar solution) for different equations in fluid dynamics, such as Euler or Boussinesq. The new numerical framework is shown to be both robust and readily adaptable to several situations. Joint work with Tristan Buckmaster, Gonzalo Cao-Labora, Ching-Yao Lai and Yongji Wang.

References

[1] Wang, Y.; Lai, C.Y.; Gómez-Serrano, J.; Buckmaster, T. *Asymptotic self-similar blow-up profile for three-dimensional axisymmetric euler equations Using Neural Networks.* Phys. Rev. Lett. 2023, 130, 244002.

15:30–15:50 Non-radial implosion for compressible Euler and Navier-Stokes in $\mathbb{T}^3$ and $\mathbb{R}^3$
Jia Shi (Massachusetts Institute of Technology, USA)

Abstract. We will discuss the smooth, non-radial solutions of the compressible Euler and Navier-Stokes equation that develop an imploding finite time singularity. The construction is motivated by the radial imploding solutions from Merle–Raphaël–Rodnianski–Szeftel (see [1],[2],[3]), and Cao-Labora–Buckmaster–Gómez-Serrano (see [4]) but is flexible enough to handle both periodic and non-radial initial data. This is a joint work with Gonzalo Cao-Labora, Javier Gómez-Serrano, and Gigliola Staffilani [5].

References

[1] F. Merle, P. Raphaël, I. Rodnianski, and J. Szeftel, *On blow up for the energy super critical defocusing nonlinear Schrödinger equations*, Invent. Math. 227.1 (2022), pp. 247–413

[2] F. Merle, P. Raphaël, I. Rodnianski, and J. Szeftel, *On the implosion of a compressible fluid I: smooth self-similar inviscid profiles*, Ann. of Math. (2) 196.2 (2022), pp. 567–778.

[3] F. Merle, P. Raphaël, I. Rodnianski, and J. Szeftel, *On the implosion of a compressible fluid II: singularity formation*, Ann. of Math. (2) 196.2 (2022), pp. 779–889

[4] T. Buckmaster, G. Cao-Labora, and J. Gómez-Serrano, *Smooth imploding solutions for 3D compressible fluids*, Arxiv preprint arXiv:2208.09445 (2022)

[5] G. Cao-Labora, J. Gómez-Serrano , J. Shi and G. Staffilani, *Non-radial implosion for compressible Euler and Navier-Stokes in* $\mathbb{T}^3$ *and* $\mathbb{R}^3$, Arxiv preprint arxiv:2310.05325 (2023)

16:00–16:20 On the effect of the Coriolis force on the enstrophy cascade

Yuri Cacchiò (Gran Sasso Science Institute, L'Aquila, ITALY)

Abstract. Geophysical fluid dynamics refers to the fluid dynamics of naturally occurring flows, such as oceans and planetary atmospheres on Earth and other planets. These flows are primarily characterized by two elements: stratification and rotation. In this article we investigate the effects of rotation on the dynamics, by neglecting stratification, in a 2D model where we incorporate the effects of the planetary rotation by adopting the β-plane approximation, which is a simple device used to represent the latitudinal variation in the vertical component of the Coriolis force.

We consider the well-known 2D β-plane Navier-Stokes equations in the statistically forced case.

Our problem addresses energy-related phenomena associated with the solution of the equations. To maintain the fluid in a turbulent state, we introduce energy into the system through a stochastic force. In the 2D case, a scaling analysis argument indicates a direct cascade of enstrophy and an inverse cascade of energy. We compare the behaviour of the direct enstrophy cascade with the 2D model lacking the Coriolis force, observing that at small scales, the enstrophy flux from larger to smaller scales remains unaffected by the planetary rotation, confirming experimental and numerical observations. In fact, this is the first mathematically rigorous study of the above equations. In particular, we provide sufficient conditions inspired by [1, 2] to prove that at small scales, in the presence of the Coriolis force, the so-called third-order structure function's asymptotics follows the third-order universal law of 2D turbulence without the Coriolis force. We also prove well-posedness and certain regularity properties necessary to obtain the mentioned results. This is a Joint work in collaboration with Amirali Hannani (Leuven) and Gigliola Staffilani (MIT).

References

[1] Bedrossian, J. and Coti Zelati, M. and Punshon-Smith, S. and Weber, F., *Sufficient conditions for dual cascade flux laws in the stochastic 2d Navier-Stokes equations.* Arch Rational Mech Anal, **237** : 103–145, (2020).

[2] Xie, J. and Bühler, O., *Exact third-order structure functions for two-dimensional turbulence.* Journal of Fluid Mechanics, **851** : 672–686, (2018).

16:30–17:00 Coffee Break

17:00–17:20 Local behaviour of non-local hypoelliptic equations

Amélie Loher (University of Cambridge, UK)

*Abstract.*We discuss local regularity properties of solutions to a class of non-local kinetic equations in divergence form. In particular, we explain why it is not clear if the Strong Harnack inequality holds for solutions of non-local equations, and how we manage to derive it. The key is to obtain a non-local-to-local bound on the tail terms that naturally affect the function values inside any local domain.

17:30–17:50 On the De Giorgi-Nash-Moser regularity theory for hypoelliptic operators

Annalaura Rebucci (Max Planck Institute for Mathematics in the Sciences, Leipzig, GERMANY)

Abstract. We present new results that extend the De Giorgi-Nash-Moser theory to hypoelliptic equations of Kolmogorov (kinetic) type with rough coefficients. A key ingredient to prove these results is a Poincaré inequality, which we derive from the construction of suitable trajectories (see [1]). The trajectories we rely on are quite flexible and allow us to consider equations with an arbitrary number of commutators and whose diffusive part is either local (second-order) or non-local (fractional order). Following [2], we later combine the Poincaré inequality with a L^2-L^∞ estimate, a Log-transformation and a classical covering argument (Ink-Spots Theorem) to deduce Harnack inequalities and Hölder regularity along the line of De Giorgi method.

References

[1] F. Anceschi, H. Dietert, J. Guerand, A. Loher, C. Mouhot, A. Rebucci, *Poincaré inequality and quantitative De Giorgi method for hypoelliptic operators*, preprint arXiv (2024).

[2] F. Anceschi, A. Rebucci, *A note on the weak regularity theory for degenerate Kolmogorov equations*, J.Diff. Eq. 341, 538–88 (2022).

July 24, 2024

11:30–12:15 Generalised Schrödinger equations
Nicola Garofalo (Università di Padova, ITALY)

Abstract. We consider the Cauchy problem for some local and nonlocal Schrödinger equations and establish some basic properties of their solutions.

12:30–12:50 Interpolative boundedness estimates for kinetic integral equations
Mirco Piccinini (Università di Pisa, ITALY)

Abstract. We investigate local properties of weak solutions to a wide class of kinetic equations where the diffusion term in velocity is an integro-differential operator, having nonnegative kernel of fractional order $s \in (0, 1)$ with merely measurable coefficients. We provide an explicit local boundedness estimate by combining together a suitable gain of integrability and a kinetic Caccioppoli-type inequality, by giving also a precise control of the long-range interactions arising from the nonlocal behaviour of the involved diffusion operator. Moreover, we will show how to deduce a similar result in the case when a p-growth conditions is assumed on the diffusion operator. This is based on a joint project in collaboration with F. Anceschi and G. Palatucci.

13:00–14:30 Lunch Break

14:30–14:15 Asymptotic average solutions and a Pizzetti-type theorem for hypoelliptic PDEs
Alessia E. Kogoj (Università di Urbino, ITALY)

Abstract. By using a Pizzetti's 1909 idea for the classical Laplacian, we introduce a notion of asymptotic average solutions. This notion enables the pointwise solvability of every Poisson equation $Lu(x) = f(x)$ with continuous data f, where L belongs to a class of hypoelliptic linear partial differential operators whose classical solutions can be characterized in terms of mean value formulae.

15:30–16:20 Lateral boundary conditions for a Kolmogorov-type PDE
Richard Sowers (Illinois University, Urbana-Champaign, USA)

Abstract. We consider a hypoelliptic Kolmogorov-type PDE motivated by a particle under the influence of a white noise force. We are interested in imposing Dirichlet conditions at a side (lateral) boundary. Specifically, we consider

$$\begin{aligned} \frac{\partial u}{\partial t}(t,x,y) &= \frac{1}{2}\frac{\partial^2 u}{\partial y^2}(t,x,y) + b(y)\frac{\partial u}{\partial x}(t,x,y) \qquad t>0,\ 0<x<\infty,\ -\infty<y<\infty \\ u(0,x,y) &= u_\circ(x,y) \qquad 0<x<\infty,\ -\infty<y<\infty \\ u(t,0,y) &= u_\partial(t,y). \qquad t>0,\ -\infty<y<\infty \end{aligned}$$

where $u_\circ$ is an initial condition and u_∂ is a lateral boundary condition. We assume that this PDE is *hypoelliptic* and that the lateral boundary is accessible. Namely, we consider functions b similar to

$$b(y) = -2 + \tanh(y);$$

these are bounded, negative, and strictly increasing in y at all points.

We construct a simple Gaussian heat kernel K inside the domain, and investigate a boundary-layer kernel K^∂. We show that this boundary layer heat kernel has a novel jump condition (stemming from a careful Laplace-type asymptotic analysis) which captures the structure of the

above PDE u near the lateral boundary. We outline a polynomial refinement of the heat kernels, and then construct a Volterra equation stemming from the variation of parameters ansatz

$$\begin{aligned} u(t,x,y) = &\int_{s=0}^{t}\int_{x_\circ>0,\,-\infty<y_\circ<\infty} K_{x_\circ,y_\circ}(t-s,x,y)\psi(s,x_\circ,y_\circ)dx_\circ\,dy_\circ\,ds \\ &+ \int_{s=0}^{t}\int_{-\infty<y_\circ<\infty} K^{\partial}_{y_\circ}(t-s,x,y)\psi^{\partial}(s,y_\circ)ds\,dy_\circ \\ &+ \int_{x_\circ>0,\,-\infty<y_\circ<\infty} K_{x_\circ,y_\circ}(t,x,y)u_\circ(x_\circ,y_\circ)dx_\circ\,dy_\circ \end{aligned}$$

where ψ and ψ^{∂} are to be found. The boundary condition corresponding to ψ^{∂} reflects the novel jump condition. We construct a Volterra equation and show (somewhat nonstandard) convergence of the Volterra equation for ψ and ψ^{∂}.

16:30–17:00 Coffee Break

17:00–17:45 Global existence for the generalized derivative NLS
Nicola Visciglia (Università di Pisa, ITALY)

Abstract In this talk, based on a joint work with M. Hayashi (Kyoto U.) and T. Ozawa (Waseda U.), we shall discuss a blow-up criterion for H^2 solutions to the generalized DNLS

$$\begin{cases} iu_t + u_{xx} + i|u|^{2\sigma}u_x = 0, \sigma > 1, (t,x) \in \mathbf{R}\times\mathbf{T} \\ u(0) = \varphi \in H^2(\mathbf{T}) \end{cases}$$

For $\sigma > 1$ the equation is no more completely integrable and hence the globalization argument is not trivial. The existence and uniquness of a local solution has been constructed in [1]. Our main result is the following blow-up alternative which improves the one described in [1].

Theorem 1. *Let $\varphi \in H^2(\mathbf{T})$. For the solution $u \in C([0,T_{max}),H^2(\mathbf{T}))$ we have the following alternative:*

(1) $T_{\max} = \infty$,
(2) $T_{\max} < \infty$ *implies* $\limsup_{t\uparrow T_{\max}} \|u(t)\|_{H^1(\mathbf{T})} = \infty$.

As a consequence of the result above and the Hamiltonian structure of the equation, we get the following global existence result for small data, which provides a positive answer to a question raised in [1].

Corollary 2. *There exists $\delta > 0$ such that if $\varphi \in H^2(\mathbf{T})$ satisfies $\|\varphi\|_{H^1(\mathbf{T})} < \delta$, then there exists a unique global solution $u \in C(\mathbf{R},H^2(\mathbf{T}))$.*

References

[1] D. M. Ambrose and G. Simpson, *Local existence theory for derivative nonlinear Schrödinger equations with noninteger power nonlinearities*, SIAM J. Math. Anal. 47 (2015), 2241–2264.

18:00–18:20 Wave kinetic theory for the forced/dissipated NLS equation
Ricardo Grande (Scuola Internazionale Superiore di Studi Avanzati, Trieste, ITALY)

Abstract. We will present some recent developments in the justification of kinetic equations in the presence of forcing and dissipation. Such settings are of particular physical relevance as they allow the study of cascades: the transfer of energy from large scales to small scales.

In this talk, we provide the first rigorous justification of such a kinetic equation in the case of a wave system governed by the cubic Schrödinger equation with a stochastic forcing and viscous dissipation. We will describe various regimes depending on the relative strength of the dissipation, the forcing and the nonlinear interactions, which give rise to different kinetic equations. Based on joint work with Zaher Hani.

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