AMS-UMI International
Joint Meeting 2024
Palermo, July 23–26, 2024

Anosov Representations and Higher Teichmüller Theory
Special Session A24

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The conference aims to bring together senior researchers and early career mathematicians from the areas of hyperbolic geometry and higher Teichmüller theory to explore the interplay between these two areas and highlight the recent breakthroughs and connections between them. The workshop will improve the exchange of ideas and the likelihood of collaboration between the United States and Europe in general. This will be facilitated also from the fact that our meeting will be preceeded by the conference "Moving to higher rank: from hyperbolic to Anosov", held in honor of Dick Canary, that will be held July 15-19, 2024 in Cetraro, Italy.

The fields of hyperbolic geometry and the more recent field of higher Teichmüller theory are major directions of current and ongoing mathematical research. Starting in the 70's, Thurston's work had a revolutionary impact on low-dimensional geometry and topology, with hyperbolic geometry playing a central role. In particular, the study of Teichmüller space, the space of (marked) complex structures on a surface S, played an important role as the moduli space of convex cocompact hyperbolic structures on 3-manifolds with boundary S. Work of Goldman, Choi and others on Thurston's (G, X)-structures generalized this to study more general geometric structures, including convex projective structures. From a completely different direction, Hitchin used a gauge theoretic approach to study the moduli space $\mathcal{R}(S, G) = \mathrm{Hom}(\pi_1(S), G)//G$ of representations of the fundamental group of a surface into higher rank groups G. In particular Hitchin showed that certain components $\mathrm{Hit}_n(S)$ of $\mathcal{R}(S, \mathbb{P}\mathrm{SL}(n, \mathbb{R}))$, now called Hitchin components, are natural generalizations of such examples. Infact, $\mathrm{Hit}_2(S)$ is the Teichmüller space $\mathrm{Teich}(S)$, and $\mathrm{Hit}_3(S)$ is the space of convex projective structures. Work of Fock and Goncharov used methods from cluster algebras and algebraic geometry to describe moduli spaces of positive representations which they named higher Teichmüller spaces and included Hitchin components.

In his seminal work, Labourie used a dynamical approach to define the notion of Anosov representations and prove many of their important properties. Anosov groups incorporate the prior examples of higher Teichmuller spaces, generalizes them, and is now a major field of study in itself, combining elements of the theory of Higgs bundles, hyperbolic geometry, the theory of Lie groups, and dynamics. The theory has been further developed in papers by Bochi-Potrie-Sambarino, Danciger-Gueritaud-Kassel, Gueritaud-Guichard-Kassel-Wienhard, Guichard-Wienhard, Guichard-Labourie-Wienhard, Kapovich-Leeb-Porti, and others. In particular this work showed that Anosov groups are the natural analogue in higher rank of convex cocompact representations into rank one Lie groups. Over the past two decades much work has shown that phenomena for cocompact hyperbolic manifolds persist for Anosov subgroups. A major theme of the conference is to explore this further and consider to what extent phenomena for cocompact hyperbolic manifolds persist for Anosov subgroups.

Schedule and Abstracts

July 23, 2024

11:00–12:00 Record breakers, slack calculus, and the Benoist limit cone
Jeff Danciger (University of Texas at Austin, USA)

Abstract. We study the boundary of the Benoist limit cone of a positive representation from a surface group into a semi-simple Lie group G, focusing on the cases $G = \mathrm{SL}(n+1, \mathbb{R})$ and $G = \mathbb{P}\mathrm{SL}(2, \mathbb{R})^n$. The limit cone, a cone in the positive Weyl chamber, is obtained by plotting the Jordan projections (essentially, the log-eigenvalues) of the elements of the group and then taking the closure of the cone spanning these points. It turns out to be convex. We investigate the question of which elements of the group (and more generally, which geodesic currents) appear on the boundary. Joint work with François Guéritaud and Fanny Kassel.

12:00–12:30 Proper affine deformations of positive representations
Neza Zager (MPIMIS Leipzing, GERMANY)

Abstract. We prove that every positive Anosov representation of a free group into $\mathrm{SO}(2n, 2n-1)$ admits a family of $\mathbb{R}^{4n-1}$-valued cocycles defining proper affine actions on $\mathbb{R}^{4n-1}$. We construct fundamental domains bounded by generalized crooked planes for these affine actions. This is joint work with Jean-Philippe Burelle.

12:30–13:00 Anosov components of triangle reflection groups in rank 2
Jennifer Vaccaro (University of Illinois, Chicago, USA)

Abstract. We seek to describe rank 2 character variety components of triangle reflection groups which contain Anosov representations, and to provide a geometric description of any Anosov boundaries. Building on the work of Lee, Lee and Stecker in $\mathrm{SL}(3, \mathbb{R})$ we observe fundamentally new phenomena: for example in $\mathrm{SL}(3, \mathbb{C})$, the Anosov component that contains the $\mathrm{SL}(3, \mathbb{R})$ Hitchin component has a nontrivial boundary. This project is ongoing, and the talk will cover recent progress and computational work.

14:30–15:00 Cubulated hyperbolic groups admit Anosov representations
Sami Douba (Institut des Hautes Études Scientifiques, FRANCE)

Abstract. While it is still not known if there is any Gromov-hyperbolic right-angled Coxeter group that cannot be realized as a convex cocompact group of isometries of some large-dimensional hyperbolic space, it is a result of Danciger–Guéritaud–Kassel that any Gromov-hyperbolic right-angled Coxeter group admits an Anosov representation. In joint work with Balthazar Fléchelles, Theodore Weisman, and Feng Zhu, we extend the latter result to all Gromov-hyperbolic quasi-convex subgroups of right-angled Coxeter groups.

15:00–15:30 Manifolds without $\mathbb{RP}^n$-structures but with Anosov representations
Lorenzo Ruffoni (Tufts University, USA)

Abstract. Surfaces admit many real projective structures. On the other hand, in dimension at least 3 there are manifolds that do not admit any real projective structure at all. Previously known examples of such "non–projective" manifolds are small, in the sense that their fundamental groups are virtually cyclic. In this talk we construct "non–projective" manifolds in every dimension at least 5 whose fundamental groups are non-elementary Gromov hyperbolic groups. While these groups do not arise from real projective geometry, they are known to admit Anosov representations because they are cubulated.

15:30–16:00 Foliations in $\mathbb{P}\mathrm{SL}(4, \mathbb{R})$ Teichmüller Theory
Alex Nolte (Rice University, USA)

Abstract. In 2008, Guichard and Wienhard carried out a beautiful study of Hitchin representations of surface groups Γ in $\mathbb{P}\mathrm{SL}(4, \mathbb{R})$ They began by singling out a domain of discontinuity Ω^1_ρ in $\mathbb{RP}^3$ for any Hitchin representation $\rho : \Gamma \to \mathbb{P}\mathrm{SL}(4, \mathbb{R})$. The motivating observation of Guichard-Wienhard's work is that Ω^1_ρ has nested $\rho(\Gamma)$-invariant foliations $\mathcal{F}_{\mathrm{pcf}}$ and $\mathcal{G}_{\mathrm{pcf}}$ whose leaves are properly convex subsets of projective planes and lines, respectively. These foliations are curious and remarkable objects. We discuss a finiteness counterpart to Guichard-Wienhard's

work: we enumerate all geometrically similar foliations of Ω^1_ρ. This proves a strong form of rigidity for Guichard-Wienhard's qualitative characterizations of projective structures associated to $\mathbb{P}\mathrm{SL}(4,\mathbb{R})$ Hitchin representations. A bit more specifically, we begin by constructing a new $\rho(\Gamma)$-invariant foliation $\mathcal{G}_{\mathrm{tcf}}$ of Ω^1_ρ whose leaves are properly convex subsets of projective lines. Our main theorem is that these are all such foliations: let $\rho : \Gamma \to \mathbb{P}\mathrm{SL}(4,\mathbb{R})$ be Hitchin, then $\mathcal{G}_{\mathrm{pcf}}$ and $\mathcal{G}_{\mathrm{tcf}}$ are the only $\rho(\Gamma)$-invariant foliations of Ω^1_ρ by properly embedded projective line segments. $\mathcal{F}_{\mathrm{pcf}}$ is the unique foliation of Ω^1_ρ by properly embedded convex domains in projective planes. Our proof involves developing a detailed general picture for the structure of the domain of discontinuity Ω^1_ρ for ρ that we will explain.

16:00–16:30 Finite-sided Dirichlet domains for Anosov representations
Colin Davalo (University of Heidelberg, MPIMIS Leipzig, GERMANY)

Abstract. Dirichlet domains provide polyhedral fundamental domains in hyperbolic space for discrete subgroups of the isometry group. For geometrically finite subgroups these domains are finite-sided, and for convex-cocompact subgroups these domains are finite sided in a stronger sense: they also define a compact fundamental domain on some open domain of discontinuity in the compactification of the hyperbolic space. Selberg introduced a similar construction of a polyhedral fundamental domain for the action of discrete subgroups of the higher rank Lie group $\mathrm{SL}(n,\mathbb{R})$ on the projective model of the associated symmetric space. His motivation was to study uniform lattices, for which these domains are finite-sided. However these domains can also be studied for smaller subgroups, and we will consider the following question asked by Kapovich: for which Anosov subgroups are these domains finite-sided? Anosov subgroups are hyperbolic discrete subgroups satisfying strong dynamical properties, but are not lattices in higher rank. We show an example of an Anosov subgroup for which the fundamental domain constructed by Selberg has infinitely many sides and provide a sufficient condition on the limit cone of an Anosov subgroup to ensure that the domain is finitely sided in a strong sense. Our techniques generalize to give a sufficient condition for an Anosov subgroup to admit finite-sided Dirichlet domains for certain Finsler metrics on the associated symmetric space.

17:00–17:30 Hausdorff dimension of hyperconvex representations of surface groups
Gabriele Viaggi (University of Rome Sapienza, ITALY)

Abstract. A discrete and faithful representation of a surface group in $\mathbb{P}\mathrm{SL}(2,\mathbb{C})$ is said to be quasi-Fuchsian when it preserves a Jordan curve on the Riemann sphere. These objects lie at the intersection of several areas of mathematics and have been studied (for example) using complex dynamics, Teichmüller theory, and 3-dimensional hyperbolic geometry. From a dynamical perspective, an important invariant of such representations is the Hausdorff dimension of the invariant Jordan curves (typically a very fractal object). It is elementary to see that this number is always at least 1. A celebrated result of Bowen establishes it is equal to 1 if and only if the quasi-Fuchsian representation is Fuchsian, that is, it is conjugate in $\mathbb{P}\mathrm{SL}(2,\mathbb{R})$. I will first describe this classical picture and then report on recent joint work with James Farre and Beatrice Pozzetti where we prove a generalization of Bowen's result for the much larger class of hyperconvex representations of surface groups in $\mathbb{P}\mathrm{SL}(d,\mathbb{C})$ (where d is arbitrary).

17:30–18:00 Divergent extended geometrically finite representations via flows
Tianqi Wang (National University of Singapore, SINGAPORE)

Abstract. The notion of extended geometrically finite representations introduced by Weisman generalizes Anosov representations by studying the convergence dynamics on group boundary extensions. We prove that divergent, extended geometrically finite representations can be interpreted as admitting dominated splitting over certain flow spaces (restricted Anosov representations in the sense of Tholozan–Wang). In particular, the example constructed by Tholozan-Wang provides a divergent geometrically finite representation.

July 24, 2024

11:30–12:30 On the boundary of Higher Teichmüller spaces
Anne Parreau (University of Grenoble-Alpes, FRANCE)

Abstract. Thurston's length compactification of the classical Teichmüller space can be generalized togeneral character varieties of finitely generated groups, using Weyl-chamber valued length functions. The boundary points can be interpreted using representations over non archimedean real closed fields, acting on real affine buildings. This provides compactifications for the higher Teichmüller spaces. I will present some results on the structure of the boundary based on joint work with Marc Burger, Alessandra Iozzi and Beatrice Pozzetti.

12:30–13:00 Hilbert geometry over non-Archimedean ordered fields
Xenia Flamm (Institut des Hautes Études Scientifiques, FRANCE)

Abstract. Convex projective geometry is a rich subject and provides an important generalisation of Riemannian geometry. Convex projective surfaces arise as a geometric interpretation of Hitchin representations in $\mathrm{SL}(3,\mathbb{R})$. Their Hilbert metric encodes important information about the representation. Understanding degenerations of convex projective structures on a surface naturally leads to the study of the Hilbert geometry of subsets of the projective plane over a non-Archimedean ordered field $\mathbb{F}$. The goal of this talk is to introduce the Hilbert metric (over $\mathbb{F}$) and to describe the metric spaces associated to convex polygons in $\mathbb{FP}^2$ endowed with the Hilbert metric. This is joint work with Anne Parreau.

14:30–15:30 Rigidity of circle packing on projective surfaces
Francesco Bonsante (University of Pavia, ITALY)

Abstract. Observing that the notion of disk in $\mathbb{CP}^1$ is invariant under projective transformations, Kojima, Mizushima and Tan proposed the study of circle packings on surfaces equipped with complex projective structure. The main observation is that combinatorially a circle packing is described by a triangulation of the surface, called the nerve of the circle packing. The Andreev-Koebe-Thurston theorem states that for a fixed triangulation T, there exists a unique Fuchsian projective structure carrying a circle packing with nerve T. Motivated by this result Kojima, Mizushima and Tan conjectured that the space of projective structures carrying a circle packing with nerve T is a section of the natural projection map of the space of projective structures to the Teichmüller space. In the talk I will explain some recent developments around this conjecture obtained in collaboration with Michael Wolf. In particular we prove that the space of projective structures carrying a circle packing with nerve T is an immersed submanifold of dimension $6g-6$ of the space of projective structures, and we prove a local projective rigdity: a circle packing cannot infinitesimally deformed within a fixed projective structure. If time permits, I will also discuss some results around conformal rigidity.

15:30–16:00 TBA
Suzanne Schlich (University of Grenoble-Alpes, FRANCE)

Abstract. TBA

16:00–16:30 Branched bending in hyperbolic 3-manifolds
Casandra Monroe (University of Texas at Austin, USA)

Abstract. A conjecture of Menasco and Reid states that a hyperbolic knot complement does not contain a closed embedded totally geodesic surface. One heuristic that is used to study the absence of such a submanifold is parabolic cohomology–in particular, if the parabolic cohomology is known to vanish in specific settings, then it serves as an obstruction to bending along such a totally geodesic hypersurface. In this talk, we consider the Borromean rings complement, which is known to not admit any closed embedded totally geodesic surfaces but still has interesting parabolic cohomology. We construct a complex of surfaces that can be used to explain these deformations.

17:00–17:30 CMC hypersurfaces in Anti-de Sitter space
Enrico Trebeschi (University of Pavia, ITALY)

Abstract. The Anti-de Sitter space is the Lorentzian analogue of the hyperbolic space, namely it is the model for negatively curved manifolds in signature $(n, 1)$. As its Riemannian counterpart, it comes with a conformal asymptotic boundary. The classical asymptotic Plateau problem in the the Anti-de Sitter space consists in finding maximal hypersurfaces (i.e. with zero mean curvature) (CMC) with a prescribed boundary at infinity. This problem has first been studied for in the $(2 + 1)$-dimensional case, because it is linked to (universal) Teichmüller theory, followed by similar results. In this talk, we show that there exists a unique hypersurface having constant mean curvature $H \in \mathbb{R}$, for any suitable prescribed boundary data. Let Λ be a non-negative $(n-1)$-topological sphere in the boundary of $\mathbb{H}^{n,1}$. For any $H \in \mathbb{R}$, there exists a unique properly embedded spacelike hypersurface Σ with constant mean curvature H and with asymptotic boundary Λ. Furthermore, all the hypersurfaces mentioned above are complete: this result extends the Cheng-Yau theorem from the flat case (Minkowski space) to the negative constant sectional curvature case. If there is time left, we will introduce some preliminary results of an on-going projects, whose goal is to estimate the geometry of a CMC-hypersurface by studying a suitable generalization of the convex hull of its asymptotic boundary. In the $(2 + 1)$-dimensional case, CMC-surfaces induce a special class of quasi-conformal maps on the hyperbolic plane, called θ-landslides. The estimates on the geometry of a CMC-surface allows to bound the maximal dilation of the corresponding θ-landslide with respect to the cross-ratio norm of its extension to the asymptotic boundary of $\mathbb{H}^2$.

17:30–18:00 High energy estimates and the Labourie conjecture
Peter Smillie (MPIMIS Leipzing, GERMANY)

Abstract. High energy harmonic maps to symmetric spaces look almost everywhere like harmonic maps into flats. I will first describe how we used a theorem of Mochizuki to that effect in order to find unstable minimal surfaces in locally symmetric spaces of rank at least three, thereby disproving the remaining cases of a conjecture of Labourie. This will then motivate some recent work removing the 'generically regular semisimple' hypothesis from Mochizuki's theorem. This is all joint work with Nathaniel Sagman.

18:00–18:30 Boundaries of Hitchin components, and things they parametrize
Charles Reid (University of Texas at Austin)

Abstract. Thurston's boundary of Teichmuller space parametrizes measured laminations, which in turn can be interpreted as instructions for building real trees. We will explain how a generalization of the Thurston boundary to the $\mathrm{SL}(n, \mathbb{R})$ Hitchin component–the spectral radius compactification–parametrizes geodesic currents which satisfy certain tropical relations. One of these tropical rank-n currents can be interpreted as instructions for building a polyhedral metric space of dimension $n - 1$. This space is in some ways more manageable than the real buildings traditionally playing this role, especially for $\mathrm{SL}(3, \mathbb{R})$ where often it is simply the universal cover of S, equipped with a triangular Finsler metric

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