

Computational Topology: Foundations, Algorithms, and Applications

Special Session A27

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“Computational Topology: Foundations, Algorithms, and Applications” is a special session scheduled for July 23-24 within the 2nd edition of the “AMS-UMI International Joint Meeting”.

Computational topology is by now a well-established field at the crossover of topology and computational geometry. It aims to transfer the power of topology for qualitative analysis to the setting of discrete approximations. Because finite data sets may be sampled from continuous underlying objects, computational topology involves not only discrete objects (such as pixelized images or finite meshes) but also an analysis of the transition between continuous and discrete. The development of this field has followed three distinct, but tightly intertwined directions: the investigation of the mathematical foundations of the subject, the quest for creating higher-performance algorithms, and the paths for new applications that those tools have opened.

The goal of this special session is two-fold. On the one hand, we want to valorize those three directions for their own importance. On the other hand, we intend to bring together well-established researchers and early-career mathematicians, enhancing the exchange of ideas and promoting collaborations between different communities.

For more information visit the following webpage:
<https://sites.google.com/view/ams-umi-computationaltopology/home-page>.

Schedule and Abstracts

July 23, 2024

11:00–11:20 Welcome to the computational topology special session

Nicolò Zava (ISTA - Institute of Science and Technology Austria, AUSTRIA)

Abstract. This talk is an introduction to computational topology, by now a well-established field at the crossover of topology and computational geometry. We will provide a general overview of the topic and the three intertwined main directions explored during the special session. We will discuss how they complement and motivate each other and connect with different fields.

This presentation is aimed mainly at researchers outside the field to, hopefully, interest curious researchers and promote proficuous exchanges between different communities.

11:30–11:50 Topology of random 2-dimensional cubical complexes

Érika Roldán Roa (Max Planck Institute for Mathematics in the Sciences and ScaDS.AI Leipzig University, GERMANY)

Abstract. We study a natural model of random 2-dimensional cubical complexes which are subcomplexes of an n -dimensional cube, and where every possible square (2-face) is included independently with probability p . Our main result exhibits a sharp threshold $p = 1/2$ for homology vanishing as the dimension n goes to infinity. This is a 2-dimensional analogue of the Burtin and Erdős-Spencer theorems characterizing the connectivity threshold for random graphs on the 1-skeleton of the n -dimensional cube. Our main result can also be seen as a cubical

counterpart to the Linial-Meshulam theorem for random 2-dimensional simplicial complexes. However, the models exhibit strikingly different behaviors. We show that if $p > 1 - \sqrt{1/2} \approx 0.2929$, then with high probability the fundamental group is a free group with one generator for every maximal 1-dimensional face. As a corollary, homology vanishing and simple connectivity have the same threshold.

References

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12:00–12:20 Variations on a barcode algorithm

Barbara Giunti (SUNY - University at Albany, USA)

Abstract. In this talk, I will provide an introduction to the standard barcode algorithm and an overview of some of its variants. I will explain what the pivot pairing is and why it provides the barcode, and how it behaves together with the two classical optimizations called clear and compress. I provide some insight into how the sparseness of the matrix can affect efficiency, and discuss the worst-case and average complexity for some special class of inputs. Finally, I will show how to update the reduced matrix so that it encodes the barcode of the original simplicial complex after the removal of some simplices from the latter.

12:30–12:50 Learning and not learning with topology

Francesco Vaccarino (Polytechnic University of Turin, ITALY)

Abstract. We present two recent results based on the use of algebraic topology and TDA in machine learning: one is the introduction of a novel loss for image processing, that leverages on Morse theory and Persistence; the other is the finding of a topological obstruction to the training of MLP.

14:30–14:50 Capturing Robust Topology in Data

Bei Wang (University of Utah, USA)

Abstract. A number of topological descriptors such as merge trees, contour trees, Reeb graphs, and persistence barcodes have been widely used in topological data analysis and visualization. However, a key question that deserves more attention is: **how can we capture robust topology in data by constructing topological descriptors that are robust to noise and outliers?**

A Reeb graph is a graphical representation of a scalar function on a topological space that encodes the topology of the level sets. A Reeb space is a generalization of the Reeb graph to a multiparameter function. We propose novel constructions of Reeb graphs and Reeb spaces that incorporate the use of a measure. Specifically, we introduce measure-theoretic Reeb graphs and Reeb spaces when the domain or the range is modeled as a metric measure space (i.e., a metric space equipped with a measure). Our main goal is to enhance the robustness of the Reeb graph and Reeb space in representing the topological features of a scalar field while accounting for the distribution of the measure. We first introduce a Reeb graph with local smoothing and prove its stability with respect to the interleaving distance. We then prove the stability of a Reeb graph of a metric measure space with respect to the measure, defined using the distance to a measure or the kernel distance to a measure, respectively. Our measure-theoretic approach allows Reeb graphs to capture robust topology in data, in line with recent advances in building robust topological descriptors.

15:00–15:20 Apparent pairs in computational topology

Ulrich Bauer (Technical University of Munich, GERMANY)

Abstract. Apparent pairs (also known as evident, shallow, close, steepness, Pareto, or minimal pairs) are a fundamental construction at the interface of persistent homology and discrete Morse theory. I will discuss their role in the context of algorithmic and computational topology, with connections to the efficient computation of persistent homology, to the geometry of Gromov-hyperbolic spaces, and to shape reconstruction from point clouds.

References

- [1] U. Bauer, *Ripser: efficient computation of Vietoris–Rips persistence barcodes*. J Appl. and Comput. Topology 5, 391–423, 2021. doi:10.1007/s41468-021-00071-5
- [2] U. Bauer, F. Roll, *Gromov hyperbolicity, geodesic defect, and apparent pairs in Vietoris–Rips filtrations*. Symposium on Computational Geometry, 2022. doi:10.4230/LIPIcs.SoCG.2022.15
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15:30–15:50 Topological Machine Learning: Applications to Raman Spectroscopy Maria Antonietta Pascali (Institute of Information Science and Technologies - CNR, ITALY)

Abstract. The advent of *machine* and *deep learning* has led to huge advancements in computer vision and data analysis, enabling a shift from handcrafted features to the automatic extraction of meaningful features through *representation learning*.

On the other hand, topological invariants offer shape informative and computable descriptors suitable for differentiating spaces; unfortunately, when applied to real-world data, these descriptors might seem too rigid. Thanks to the theory of *persistent homology* (PH), it is possible to use them to conduct intrinsically multiscale analysis.

Merging PH and machine learning methods allowed us to design and develop a *Topological Machine Learning (TML)* pipeline [1], leveraging the informative content of topological features which are different and complementary to those used in deep learning architectures, such as convolutional neural networks. Such a pipeline that associates persistence diagrams to digital data, via the most appropriate filtration for the type of data considered. Using a grid search approach, representation methods and parameters optimal for the classification task assigned are determined. We assessed the performance of our pipeline, and in parallel, we compared the different representation methods, on popular benchmark datasets, showing promising results.

In real-world problems, the pipeline has been exploited in the medical domain for a challenging task: the classification of Raman spectroscopy (RS).

RS is based on evaluating the inelastic scattering process in which photons incident on a sample transfer energy to or from molecular vibrational modes. Such information is stored in a spectrum. Instead of focusing the analysis on predetermined peaks or windows in the spectra, current research considers the Raman spectrum as a biochemical signature of the sample. In the biomedical domain (e.g., histopathology and oncology), RS represents a fast and efficient diagnostic tool that has found applications to several kinds of biological samples, including cellular tissues, cell lines and fluids, providing practical tools for assessing disease presence and grade. In this context, the TML pipeline achieved convincing results for the grading of chondrosarcoma [2] while in [3,4] it has been used to distinguish the Alzheimer’s disease from other neurodegenerative pathologies (Bando Salute 2018 PRAMA project co-funded by the Tuscany Region). The present contribution will include the description and discussion of strengths and limitations of TML methods applied to the analysis and classification of data from RS in the biomedical domain.

References

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- [2] F. Conti, M. D’Acunto, C. Caudai, S. Colantonio, R. Gaeta, D. Moroni, M. A. Pascali, *Raman spectroscopy and topological machine learning for cancer grading*, Scientific Reports, vol. 13 (2023), no. 1, 7282.
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16:00–16:20 Random Simple Homotopy Theory

Bruno Benedetti (University of Miami, USA)

Abstract. We implement an algorithm RSHT (Random simple-homotopy) to study the simple-homotopy types of simplicial complexes. The algorithm combines elementary simplicial collapses with pure elementary expansions, and it represents a valide alternative to Discrete Morse Theory, with a particular focus on contractible spaces and on substructures within higher-dimensional complexes. This is joint work with Crystal Lai, Davide Lofano, and the late Frank Hagen Lutz, to whose memory the talk is dedicated.

17:00–17:20 Discrete Morse theory for open complexes

Nicholas Scoville (Ursinus College, USA)

Abstract. In this talk, we develop a discrete Morse theory for open simplicial complexes $K = X \setminus T$ where X is a simplicial complex and T a subcomplex of X . A discrete Morse function f on K gives rise to a discrete Morse function on the order complex of K , and the topology change determined by f on K can be understood by analyzing the topology change determined by the discrete Morse function on the order complex. This topology change is given by a structure theorem on the level subcomplexes of the order complex. Finally, we show that the Borel-Moore homology of K , a homology theory for locally compact spaces, is isomorphic to the homology induced by a gradient vector field on K and deduce corresponding weak Morse inequalities.

17:30–17:50 Topological Data Analysis of Knowledge Networks

Lori Ziegelmeier (Macalester College, USA)

Abstract. Knowledge networks can organize complex data by constructing graphs where nodes are concepts or ideas and edges represent connections of significance. Understanding the structure of these knowledge networks to uncover how science progresses over time is of interest to researchers studying the “Science of Science.” In this project, we are interested in understanding cycles or holes within a network, which can be thought of as gaps in knowledge. We use topological data analysis, and in particular, persistent homology filtered through time where the nodes represent scientific concepts and edges between two nodes are added at the time when they appear together in an abstract of a scientific paper. We study properties of these knowledge gaps in multiple dimensions such as when they form, when they no longer remain, and the concepts and papers that make up the cycles. We observe that papers involved in the knowledge gaps are cited more frequently than papers that are not.

July 24, 2024

11:30–11:50 Max-Flow for Circular Mapper Graphs

Matteo Pegoraro (Aalborg University, USA)

Abstract. We start from the problem of analysing sets with periodic boundary conditions, which often arise in materials science, and develop a notion of circular max-flow for mapper graphs with a map into the circumference. Roughly speaking, we compute max-flow slicing the graph with the fibers of the map into the circle. We study theoretic and computational properties of such definition, which we then test on simulated data representing glass structures, finding very high correlation with the conductivity of such materials.

12:00–12:20 Topology meets signal processing and learning

Stefania Sardellitti (Universitas Mercatorum, Rome, ITALY)

Abstract. In recent years, the ever-growing interest in machine learning and in signal processing communities for data processing and analysis has driven the development of advanced models and tools to encode the interdependencies among data entities. Within this context, topological spaces have emerged as the ideal domain for capturing and encoding the complex relationships

among data entities. As a result, Graph Signal Processing (GSP) has risen as a powerful branch of signal processing for analyzing and processing signals defined over the nodes of graphs, which are simple topological spaces able to represent pairwise relations between data.

However, in many applications such as social and communication networks, brain and biological networks, the complex interactions among data must be expressed through multiway relations that cannot be represented using simple dyadic relations as with graphs. To overcome this limitation, the more general framework Topological Signal Processing (TSP) has recently emerged as a fascinating area of research that merges concepts from algebraic topology with signal processing to represent and analyse signals defined over higher-order topological spaces, such as simplicial and cell complexes, which are capable of encoding multiway relations among data.

This talk aims to show how cross-fertilization across diverse fields such as topology, signal processing, and machine learning, not only enriches our theoretical understanding but also enhances practical applications, paving the way for breakthroughs in tackling the intricate challenges of data-driven learning.

I will first introduce basic graph signal processing tools and algorithms for inferring the graph topology from data, with applications to brain networks. Then, I will present the topological signal processing framework for the analysis and processing of signals defined over higher order topological structures such as simplicial complexes, cell complexes and a novel class of generalized cell complexes, named planar hollow cell complexes. We will discuss efficient methods for sampling and recovering edge signals, filtering signals, and strategies for topology inference from data. We quantify the advantages of using topological signal processing for various applications such as anomalous traffic detection, recovering of real data traffic flows and image segmentation.

12:30–12:50 Tagged barcodes for the topological analysis of gradient-like vector fields **Clemens Bannwart (University of Modena and Reggio Emilia, ITALY)**

Abstract. In topological data analysis, persistent homology studies a continuous function $f: M \rightarrow \mathbb{R}$ by filtering M by the sublevel sets of f and then applying homology in order to obtain a persistence barcode. Our goal is to develop a persistence theory for gradient-like vector fields on Riemannian manifolds. Such a vector field may not be the gradient of any function, so a difficulty is that there is no canonical filtration of M . Inspired by some recent papers, we are decomposing parametrized chain complexes, but with epimorphic rather than the usual monomorphic internal maps that come from filtrations.

In line with this program, our first contribution is the study of the category of *tame epimorphic parametrized chain complexes*, i.e. functors that take real values to chain complexes, with internal maps always epimorphic and that may fail to be isomorphisms at most at finitely many times. We show that objects in this category can be decomposed into simple direct summands enumerated by a multiset of tagged real intervals, thus yielding a *tagged barcode*. After extending the standard interleaving and bottleneck distances to this setting, we prove an isometry theorem stating that the interleaving distance between two tame epimorphic parametrized chain complexes is equal to the bottleneck distance between their tagged barcodes.

As a second contribution we give a general procedure that, under some genericity conditions, starts from a chain complex with chosen bases and weights on them, and constructs a tame epimorphic parametrized chain complex.

Having this construction at our disposal, we are in the position of associating, in a stable way, a barcode of tagged intervals with any generic enough smooth vector field (precisely, Morse-Smale with pairwise different distances between singular points), without closed orbits (i.e. gradient-like). Given such a vector field v on a Riemannian manifold M , we apply our construction to its Morse complex, viewed as a weighted based chain complex with bases and weights given by the singular points of v and distances between them, respectively. We show that the map taking such a vector field to its tagged barcode is continuous, thus proving stability.

Thanks to the generality of the approach, we can also apply it to combinatorial gradient-like vector fields. Our third contribution is that the tagged barcode of a smooth vector field on a compact Riemannian manifold can be approximated arbitrarily well, in terms of bottleneck

distance, by the tagged barcodes of combinatorial vector fields defined on sufficiently refined triangulations of the manifold.

The full version of this work is available at [1].

References

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14:30–14:50 Grounded persistent path homology: a stable, topological descriptor for weighted digraphs

Thomas Chaplin (University of Oxford, UNITED KINGDOM)

Abstract. Weighted digraphs are used to model a variety of natural systems and can exhibit interesting structure across a range of scales. In order to understand and compare these systems, we require stable, interpretable, multiscale descriptors. To this end, we propose grounded persistent path homology (GRPPH) – a new, functorial, topological descriptor that describes the structure of an edge-weighted digraph via a persistence barcode. We show there is a choice of circuit basis for the graph which yields geometrically interpretable representatives for the features in the barcode. Using the theory of path homotopy, we show the barcode is stable, in bottleneck distance, to both numerical and structural perturbations. We compute GRPPH on a range of model vascular flow networks and verify that the descriptor can distinguish between different architectures and boundary conditions, with interpretable features explaining the distinctions. Moreover, in order to simulate pathological networks, we empirically investigate the behaviour of the descriptor under random edge removal and rewiring.

15:00–15:20 Using Directed Topology to Understand 2-Parameter Persistence

Robyn Brooks (University of Utah, USA)

Abstract. Unlike single-parameter persistent homology, which is completely described by a persistence diagram, multi-parameter persistence modules contain more information than it is possible to handle, understand, and visualize easily. Even bi-persistence modules can be quite complicated, although there exists a growing number of visualization tools and methods for computing bi-persistence invariants. The overall goal of this talk is to introduce a new construction relating the parameter space of a 2-parameter persistence module to a directed space, with the purpose of leveraging tools from directed topology to visualize and compute related invariants.

In this talk, I will introduce directed Euclidean cubical complexes and the associated definition of directed homotopy within these spaces. I will give some examples and share some results on *directed collapse*, a way of collapsing cells of a directed Euclidean cubical complex which preserves the relevant spaces of directed paths in the original complex.

I will then introduce a new construction which uses directed Euclidean cubical complexes to model the parameter space of a 2-parameter persistence module. One goal of this construction is to relate the set of 1-d persistence diagrams along non-decreasing paths for a particular 2-parameter persistence module to the space of directed paths in the parameter space of that module. From this construction, we can leverage the tool of directed collapse to simplify the parameter space while still maintaining all necessary information to understand the module.

15:30–15:50 On the use of the extended Pareto grid in 2-parameter persistent homology

Patrizio Frosini (University of Bologna, ITALY)

Abstract. In recent years, the concept of *extended Pareto grid* has been introduced as a tool to study biparametric persistence for regular functions defined on smooth closed manifolds. Given a sufficiently regular function $\varphi = (\varphi_1, \varphi_2)$ from a smooth closed manifold M to \mathbb{R}^2 , let us assume that $\{p_1, \dots, p_r\}$ and $\{q_1, \dots, q_s\}$ are the sets of critical points of φ_1 and φ_2 , respectively. The extended Pareto grid of φ is the subset $\Gamma(\varphi)$ of \mathbb{R}^2 defined as the union $A \cup B \cup C$ with

- (1) $A = \{\varphi(p) \in \mathbb{R}^2 : p \in M, \nabla\varphi_1(p) \text{ and } \nabla\varphi_2(p) \text{ are parallel, and } \nabla\varphi_1(p) \cdot \nabla\varphi_2(p) \leq 0\}$;
- (2) $B = \bigcup_{i=1}^r v_i$ where v_i is the vertical half-line $\{(x, y) \in \mathbb{R}^2 \mid x = \varphi_1(p_i), y \geq \varphi_2(p_i)\}$;
- (3) $C = \bigcup_{j=1}^s h_j$ where h_j is the horizontal half-line $\{(x, y) \in \mathbb{R}^2 \mid x \geq \varphi_1(q_j), y = \varphi_2(q_j)\}$.

For every $(a, b) \in]0, 1[\times \mathbb{R}$ and $p \in M$, let us set

$$\varphi_{(a,b)}^*(p) = \min\{a, 1-a\} \max\left\{\frac{\varphi_1(p) - b}{a}, \frac{\varphi_2(p) + b}{1-a}\right\}$$

and consider the line $r_{(a,b)} = \{(at + b, (1-a)t - b) : t \in \mathbb{R}\}$.

Given two functions $\varphi, \psi : M \rightarrow \mathbb{R}^2$, the *matching distance* D_{match} [1] is defined by setting

$$D_{\text{match}}(\varphi, \psi) = \sup_{(a,b) \in]0, 1[\times \mathbb{R}} d_B\left(\text{Dgm}\left(\varphi_{(a,b)}^*\right), \text{Dgm}\left(\psi_{(a,b)}^*\right)\right)$$

where d_B is the bottleneck distance between persistence diagrams.

The following result and the stability of persistence diagrams allow us to follow the points of $\text{Dgm}\left(\varphi_{(a,b)}^*\right)$ while the pair (a, b) changes.

Theorem 1. *If $(a, b) \in]0, 1[\times \mathbb{R}$, $(u, v) \in \text{Dgm}\left(\varphi_{(a,b)}^*\right)$, $u \neq v$, $w \in \{u, v\}$ and $w \neq \infty$, then a point (x, y) in $r_{(a,b)} \cap \Gamma(\varphi)$ exists such that $w = \frac{\min\{a, 1-a\}}{a}(x - b) = \frac{\min\{a, 1-a\}}{1-a}(y + b)$.*

In this talk, we will illustrate how Theorem 1 can be used to prove some properties of biparametric persistence [2, 3].

References

- [1] Andrea Cerri, Barbara Di Fabio, Massimo Ferri, Patrizio Frosini, Claudia Landi, *Betti numbers in multidimensional persistent homology are stable functions*, *Mathematical Methods in the Applied Sciences*, 36 (2013), 1543–1557.
- [2] Andrea Cerri, Marc Ethier, Patrizio Frosini, *On the geometrical properties of the coherent matching distance in 2D persistent homology*, *Journal of Applied and Computational Topology*, 3 (2019), 381–422.
- [3] Marc Ethier, Patrizio Frosini, Nicola Quercioli, Francesca Tombari, *Geometry of the matching distance for 2D filtering functions*, *Journal of Applied and Computational Topology*, 7 (2023), 815–830.

16:00–16:20 Computational complexity in multiparameter persistence

Håvard Bakke Bjerkevik (SUNY - University at Albany, USA)

Abstract. For topological data analysis to be useful, we need efficient algorithms for computation. Our hunt for computational efficiency involves understanding which problems are NP-hard, and which problems might allow polynomial time algorithms. It is known that computing the interleaving distance for merge trees and multiparameter modules is NP-hard, and in recent work, Magnus Botnan and I proved that NP-hardness also holds in these cases for ℓ^p -versions of the interleaving distances. I will discuss these results and related classes of problems for which the computational complexity is still poorly understood. This includes an important meta-question in multiparameter persistence; namely, how much information do we have to sacrifice to turn a hard problem into a tractable one?

References

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17:00–17:20 DREiMac: Dimensionality Reduction with Eilenberg-MacLane Coordinates

Jose Perea (Northeastern University, USA)

Abstract. Dimensionality reduction is the machine learning problem of taking a data set whose elements are described with potentially many features (e.g., the pixels in an image), and computing representations which are as economical as possible (i.e., with few coordinates). In this talk, I will present a framework to leverage the topological structure of data (measured via persistent cohomology) and construct low dimensional coordinates in classifying spaces consistent with the underlying data topology.

References

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17:30–17:50 Koszul complexes and generalized persistence

Francesca Tombari (Max Planck Institute for Mathematics in the Sciences, Leipzig, GERMANY)

Abstract. Persistence modules are commonly encoded as functors from a poset to the category of finite dimensional vector spaces. In the case of one-parameter persistence, for example, the indexing poset can be restricted, under certain assumptions, to a finite subset of \mathbb{R} , and, in the case of multi-parameter persistence, to a finite subset of \mathbb{R}^k , for $k > 1$. A lot of research has been conducted to infer invariants for persistence modules. In this talk, we concentrate on the Betti numbers, homological algebra invariants coming from minimal free resolutions of the persistence modules. In particular, we show that, under certain assumptions on the indexing poset, it is possible to make use of Koszul complexes to determine the Betti numbers of the persistence module, without constructing its entire minimal free resolution. Koszul complexes are chain complexes that can be obtained just by evaluating the persistence module on certain elements of the poset. Thus, in a way, they just depend on the local structure of the indexing poset. After having established how to obtain Betti numbers from Koszul complexes and under which hypothesis this can be done, we show how to translate these methods from standard to relative homological algebra. In particular, we show how to use Koszul complexes, defined in standard homological algebra, to compute relative Betti numbers of persistence modules.

References

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18:00–18:20 Mapping spaces of persistence diagram into the Hilbert space with controlled distortion

Žiga Virk (University of Ljubljana and Institute IMFM, SLOVENIA)

Abstract. Stability is one of the most important properties of persistent homology. Similar inputs yield similar persistence diagrams. In this context, the space of persistence diagrams is typically equipped with the bottleneck metric. In order to apply statistical tools or further data analytic techniques to collections of persistence diagrams, we thus need to use a map from the space of persistence diagrams into a Euclidean or Hilbert space. In the past decade dozens of such maps have been proposed, including persistence landscape and persistence images. These maps are typically stable (Lipschitz). However, none of them has explicit lower bounds on distortion and hence they provide no control on the loss of information. In this talk we will present Lipschitz maps from certain spaces of persistence diagrams into Hilbert and Euclidean spaces with explicit

distortion functions. The maps are fairly geometric, consisting essentially of bottleneck distances to specific landmark diagrams, and are thus easily implementable. The idea for the construction comes from the quantification of certain classical constructions in dimension theory.

References

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