Nonlinear Problems with Nonstandard Growth Conditions and Analysis on Metric Spaces
Special Session A18

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The session focuses on the study of nonlinear partial differential equations under nonstandard growth conditions which constitute an important sub-field of the calculus of variations. Variational methods are powerful tools not only in investigating existence of solutions, but also in obtaining information on the behaviour and regularity properties of minimizers and, more generally, quasiminimizers that minimize the energy functional up to a multiplicative constant.

The topics include regularity theory for a wide class of singular and degenerate elliptic and parabolic equations as well as stability properties essential in applications of PDE. As certain natural physical settings are non-smooth, by necessity the research is conducted in the general setting of (potentially) nonsmooth metric measure spaces. One of the advantages of this kind of approach is that it embraces many different spaces, as a consequence the results can be applied in manifolds, graphs, vector fields and groups, just to mention a few. We are interested both in theoretical aspects of nonlinear partial differential equations and also in their applications to the regularity theory. In particular, we present regularity questions including boundedness and Hölder continuity of solutions and higher integrability properties of the gradients of solutions.

The nonlinear partial differential equations under study are connected to many different applications, for example diffusion in highly nonhomogeneous media and the motions of multiphased fluids in porous media.

The aim of the session is bringing together leading minds in the field and early career mathematicians in a relaxed, informal atmosphere conducive of creating new scientific collaborations. We intend to make the environment more inclusive and the participants reflect the diversity of mathematicians in the field.

The Nonlinear Problems with Nonstandard Growth Conditions and Analysis on Metric Spaces Session is scheduled on July 23-24. We look forward to seeing you in Palermo!

Schedule and Abstracts

July 23, 2024

Chair: Vincenzo Vespri

11:00–11:20 The Leray-Lions existence theorem under general growth conditions
Paolo Marcellini (University of Florence, ITALY)

Abstract. It is an edition, under more general growth conditions, of the celebrated existence theorem of weak solutions to a class of Dirichlet problems for second order nonlinear elliptic equations under the so-called natural growth conditions, published in 1965 by Jean Leray and Jacques-Louis Lions. We describe some existence and regularity results recently obtained in collaboration with G.Cupini and E.Mascolo.

11:30–11:50 Superposition of subsolutions to infinite Laplace equation in disjoint variables
Xiaodan Zhou (Okinawa Institute of Science and Technology Graduate University, JAPAN)

Abstract. It is well known that the sum of two subharmonic functions is still subharmonic. However, in general we cannot expect the same type of results to hold for nonlinear elliptic equations. On the other hand, one may easily verify a variant of this property with disjoint variables: if $u_1(x)$ is a smooth subsolution to $\Delta u = 0$ in $\mathbb{R}^m$ and $u_2(y)$ is a smooth subsolution to $\Delta u = 0$ in $\mathbb{R}^n$, then $u_1(x) + u_2(y)$ is a subsolution of $\Delta u = 0$ in $\mathbb{R}^{m+n}$. It turns out that this new superposition property can be extended to several nonlinear operators. The purpose of this talk is to show such a result for the infinity Laplace equation. We will also discuss further generalization for infinite Laplacian associated to a frame of vector fields in $\mathbb{R}^n$. The talk is based on joint work with Qing Liu and Juan Manfredi.

12:00–12:20 Elliptic systems and double phase functionals
Francesco Leonetti (University of L'Aquila, ITALY)

Abstract. We consider the system of partial differential equations in divergence form

$$\begin{cases} -\sum_{i=1}^{n} D_i \left[A_i^1(x, u(x), Du(x))\right] & = 0, \\ -\sum_{i=1}^{n} D_i \left[A_i^2(x, u(x), Du(x))\right] & = 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ -\sum_{i=1}^{n} D_i \left[A_i^m(x, u(x), Du(x))\right] & = 0, \end{cases}$$

where $x \in \Omega \subset \mathbb{R}^n$, Ω is bounded and open, $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ and $u^1, u^2, ..., u^m$ are the components of u, so that, $u = (u^1, u^2, ..., u^m)$. We assume that u is a weak solution of the previous system, $A(x, z)$ is measurable with respect to x and continuous with respect to z.

It is well known that solutions to elliptic systems may be unbounded, [2]. Nevertheless, for some special classes of systems, it can be proved that solutions are bounded. We mention a recent result of this kind: [1]. In such a paper, we are dealing with the so-called p, q-growth conditions, [3]. Starting from [4], we discuss some examples suggested by double phase functionals.

References

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[3] P. Marcellini, *Regularity and existence of solutions of elliptic equations with p,q-growth conditions*, J. Differ. Equ. 90 (1) (1991), 1–30.

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12:30–12:50 On regularity for nonstandard growth functionals on metric spaces
Antonella Nastasi (University of Palermo, ITALY)

Abstract. We present some results on a class of double phase integrals characterized by nonstandard growth conditions. The study focuses on regularity theory, specifically local and global higher integrability, for quasiminimizers of a double phase integral with (p,q)-growth. The methods are variational-based in the setting of metric measure spaces with a doubling measure and a Poincaré inequality. The main feature is an intrinsic approach to double phase Sobolev-Poincaré inequalities. The results were obtained in collaboration with Juha Kinnunen (Aalto University) and Cintia Pacchiano Camacho (Calgary University).

References

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Chair: Nageswari Shanmugalingam

14:30–14:50 Conformal transformations preserving Besov energy
Riikka Korte (Aalto University, FINLAND)

Abstract. One approach to solving Dirichlet or Neumann boundary value problems on unbounded domains is by changing the metric and the measure of the space in such a way that the p–energy is preserved and the domain becomes bounded in the new metric. After the transformation, the direct methods of calculus of variation can be applied to the transformed problem. This approach has been applied in the metric measure spaces setting for uniform domains with bounded boundary by Gibara, Korte and Shanmugalingam. In this case, it is possible to use a transformation that does not change the metric or the measure on the boundary. If we want to apply this kind of approach for problems on domains with unbounded boundary, we have to use transformations that change also the boundary and therefore we have to study also what happens to the Besov spaces – i.e. the trace space of Sobolev functions. As a first step in developing methods into this directions, we will discuss in this talk what kind of transformations preserve the Besov energy.

This is based on joint work with A. And J. Björn, S. Rogovin and T. Takala.

15:00–15:20 Dirichlet Problems on Unbounded Metric Measure Spaces
Ryan Gibara (University of Cincinnati, USA)

Abstract. In this talk, we will discuss the Dirichlet problem on an unbounded uniform domain in a metric measure space with boundary datum belonging to an appropriate Besov class that is related to the ”codimensionality” of the boundary of the domain. In the case where the boundary is itself bounded, this is accomplished via a conformal transformation that renders the domain bounded while maintaining the boundary intact, and so the known results for solving the Dirichlet problem on a bounded uniform domain can be employed. When the boundary is not bounded, this is accomplished by approximating the domain by ones with bounded boundary and applying the first case.

15:30–15:50 Comparison of Dirichlet and Newtonian Sobolev spaces
Ilmari Kangasniemi (University of Cincinnati, USA)

Abstract. We consider the following question. Suppose that (X, d, μ) is an unbounded metric measure space with $\mu(X) = \infty$. Let $D^{1,p}(X)$ be the Dirichlet space consisting of functions $f\colon X \to \mathbb{R}$ with an L^p-integrable upper gradient $\rho \in L^p(X)$, and let $N^{1,p}(X) = D^{1,p}(X) \cap L^p(X)$ be the corresponding Newtonian Sobolev space. When does one have $D^{1,p}(X) = N^{1,p}(X) + \mathbb{R}$?

We show that $D^{1,p}(X) \neq N^{1,p}(X) + \mathbb{R}$ for a relatively general class of unbounded doubling spaces X with infinite measure. We also show that for hyperbolic spaces with multiple ends, one has $D^{1,p}(X) \neq N^{1,p}(X) + \mathbb{R}$. However, in hyperbolic spaces with a single end, the question becomes more interesting; as an example of this, we discuss how in the standard n-dimensional hyperbolic space $\mathbb{H}^n$, the exponent $p = n - 1$ acts as a cut-off point for this phenomenon.

16:00–16:20 Self-improvement of fractional Hardy inequalities in metric measure spaces
Josh Kline (University of Cincinnati, USA)

Abstract. In the setting of a compact, doubling metric measure space (X, d, μ), we say that a closed set $E \subset X$ satisfies an (α, p)-Hardy inequality for $1 < p < \infty$ and $0 < \alpha < 1$ if the following holds for all Lipschitz functions u which vanish on E:

$$\int_{X \setminus E} \frac{|u(x)|^p}{\operatorname{dist}(x, E)^{\alpha p}} d\mu(x) \leq C \int_X \int_X \frac{|u(x) - u(y)|^p}{d(x, y)^{\alpha p} \mu(B(x, d(x, y)))} d\mu(y) d\mu(x).$$

In this talk, we discuss the relationship between such fractional Hardy inequalities on X and p-Hardy inequalities in the hyperbolic filling of X. By first showing that a p-Hardy inequality implies the validity of weighted versions for a certain class of *regularizable* weights, we then use the structure of the hyperbolic filling to show self-improvement of the (α, p)-Hardy inequality in both exponent p and smoothness parameter α.

July 24, 2024

Chair: Paolo Marcellini

11:30–11:50 Hölder Continuity of the Gradient of Solutions to Doubly Non-Linear Parabolic Equations

Ugo Pietro Gianazza (University of Pavia, ITALY)

Abstract. We study the local behavior of non-negative weak solutions to the doubly non-linear parabolic equation

$$\partial_t u^q - \operatorname{div}\left(|Du|^{p-2}Du\right) = 0$$

in a space-time cylinder. Hölder estimates are established for the gradient of its weak solutions in the so-called *super-critical fast diffusion regime* $0 < p-1 < q < \frac{N(p-1)}{(N-p)_+}$ where N is the space dimension. Moreover, decay estimates are obtained for weak solutions and their gradient in the vicinity of possible extinction time. Two main components towards these regularity estimates are a time-insensitive Harnack inequality that is particular about this regime, and Schauder estimates for the parabolic p-Laplace equation.

References

[1] V. Bögelein, F. Duzaar, U. Gianazza, N. Liao, C. Scheven, *Hölder Continuity of the Gradient of Solutions to Doubly Non-Linear Parabolic Equations*, arXiv preprint 2305.08539 (2023), 1–142.

12:00–12:20 A PDE-based approach to Borell-Brascamp-Lieb inequality
Qing Liu (Okinawa Institute of Science and Technology Graduate University, JAPAN)

Abstract. In this talk, we provide a new PDE perspective for the celebrated Borell-Brascamp-Lieb inequality [2][3]. In contrast to previously known proofs involving techniques from convex analysis or optimal transport, our new proof is based on properties of diffusion equations of porous medium type, particularly a generalized concavity maximum principle and large time asymptotics. Our approach reveals a deep connection between the Borell-Brascamp-Lieb inequality and nonlinear parabolic equations. An analogous idea was introduced in [1] to prove the special case of the Prékopa-Leindler inequality by using the heat equation. Our analysis serves as a nonlinear generalization and improvement of this PDE approach. Furthermore, we recover the equality condition for the Prékopa-Leindler inequality by further exploiting additional properties of the heat equation including the eventual log-concavity and backward uniqueness of solutions.

References

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12:30–12:50 Existence of parabolic minimizers to the total variation flow on metric measure spaces
Cintia Pacchiano Camacho (University of Calgary, CANADA)

Abstract. In this project we discuss some fine properties and existence of the variational solutions to the Total Variation Flow. Instead of the classical Euclidean setting, we intend to work mostly in the general setting of metric measure spaces. During the past two decades, a theory of Sobolev functions and BV functions has been developed in this abstract setting. A central motivation for developing such a theory has been the desire to unify the assumptions and methods employed in various specific spaces, such as weighted Euclidean spaces, Riemannian manifolds, Heisenberg groups, graphs, etc.

The total variation flow can be understood as a process diminishing the total variation using the gradient descent method. This idea can be reformulated using parabolic minimizers, and it gives rise to a definition of variational solutions. The advantages of the approach using a minimization formulation include much better convergence and stability properties. This is a very essential advantage as the solutions naturally lie only in the space of BV functions.

We first give an existence proof for variational solutions u associated to the total variation flow. Here, the functions being considered are defined on a metric measure space (X, d, μ). For such parabolic minimizers that coincide with a time-independent Cauchy-Dirichlet datum u_0 on the parabolic boundary of a space-time-cylinder $\Omega \times (0, T)$ with $\Omega \subset X$ an open set and $T > 0$, we prove existence in the weak parabolic function space $L^1_w(0, T; \mathrm{BV}(\Omega))$. In this paper, we generalize results from a previous work by Bögelein, Duzaar and Marcellini and argue completely on a variational level. This is a joint project with Vito Buffa and Michael Collins.

References

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[2] Buffa, V.: *BV Functions in Metric Measure Spaces: Traces and Integration by Parts Formulæ*, Ph.D. Thesis, Università degli Studi di Ferrara, 2017, available at: http://cvgmt.sns.it/paper/4110.

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Chair: Antonella Nastasi

14:30–14:50 Some results about doubly nonlinear equations
Vincenzo Vespri (University of Florence, ITALY)

Abstract. The term doubly nonlinear refers to the fact that the diffusion part depends nonlinearly both on the gradient and the solution itself. Such equations describe several physical phenomena and were introduced by Lions and Kalashnikov. These equations have an intrinsic Mathematical interest because they represent a natural bridge between the more natural generalisation of the heat equation: the p-Laplacian and the Porous Medium equations. Especially in the last few years, many papers were devoted to this topic. The idea is to give an unified approach comprehensive of the Porous Medium and the p-Laplacian equations. The approaches are sometimes not rigorous, sometimes not with sharp assumptions or with unnecessary long proofs. In this talk I will speak about the state-of-the-art, my contribution and the open questions.

15:00–15:20 Regularity and Existence Results for Doubly Nonlinear Anisotropic Diffusion Equations
Matias Vestberg (Uppsala University, SWEDEN)

Abstract. We present some recent results regarding the regularity and existence theory for doubly nonlinear anisotropic diffusion equations. Some questions that will be considered are the boundedness of solutions, expansion of the support, comparison principles and the existence of solutions to the Cauchy problem.

15:30–15:50 Fine boundary continuity for degenerate double-phase diffusion
Simone Ciani (University of Bologna, ITALY)

Abstract. We present a study on the boundary behavior of solutions to parabolic double-phase equations, through the celebrated Wiener's sufficiency criterion.

The analysis is conducted for cylindrical domains and the regularity up to the lateral boundary is shown in terms of either its p or q capacity, depending on whether the phase vanishes at the boundary or not. Eventually we obtain a fine boundary estimate that, when considering uniform geometric conditions as density or fatness, leads us to the boundary Hölder continuity of solutions.

In particular, the double-phase elicits new questions on the definition of an adapted capacity.

This is a joint work in collaboration with Eurica Henriques and Igor Skrypnik.

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