Geometric variational models with nonlocal energies
Special Session A6

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Nonlocal geometric variational models have attracted significant attention in recent years. Nonlocal interactions are ubiquitous in physical models, from material science to chemistry and biology. The pertinence of nonlocal geometric models is that they naturally introduce length scales, which then are used to investigate the geometry of microstructures in macroscopic domains. The study of such models is a very active research direction, aiming at understanding e.g.: collective behavior in biological systems; the geometry of phase transitions/singularities of steady states for a large class of PDEs; the mechanisms behind pattern formation in self-organizing systems induced by competing short-range attractive/long-range repulsive interactions. The aim of this session is to bring together leading experts from US, Italy and Europe working on complementary and interconnected problems in the field, so as to favour fruitful interactions and collaborations.

For more information visit `https://sites.google.com/view/ams-umi-geom-var/`.

Schedule and Abstracts

July 23, 2024

11:00–11:45 Equilibrium shapes of liquid drops with discrete charges
Cyrill Muratov (University of Pisa, ITALY)

Abstract. In this talk I will present our treatment of a geometric variational problem arising from modeling the equilibrium shapes of liquid drops whose energy presents a competition of surface tension with the repulsive Coulombic energy of a fixed number of point charges inside the drop. The continuum analog of this problem in which the liquid is treated as a perfect conductor is known to be variationally ill-posed, hence the discrete nature of the charges preserved in our model presents a non-trivial regularization whose properties are far from obvious. In our model, we make a simplification of no dielectric contrast between the liquid and its surroundings, which nevertheless is an appropriate assumption for charged drops of liquid helium that are used in applications to quantum chemistry. For large numbers of charges, we identify a sharp charge threshold as the volume of the drop goes to infinity jointly with the number of charges. This threshold separates the regime of existence of minimizers from that of non-existence and turns out to be considerably lower than the one predicted by Rayleigh for continuum charge distributions, and below the threshold the minimizer looks like a small perturbation of a ball with charges distributed approximately uniformly over the drop surface. Above the threshold, on the other hand, it is always convenient to evaporate a single charge from the drop and move it to infinity to lower energy.

12–12:20 Energy driven pattern formation for local/non-local systems: the isotropic and crystalline case
Eris Runa (Gran Sasso Science Institute, ITALY)

Abstract.

In this talk I will discuss symmetry breaking and pattern formation for a family of functionals with local/non-local interactions in general dimension. Such functionals arise naturally in applications and contain a local term penalizing interfaces, and a non-local term favouring oscillations. In most physical situations such functionals have either isotropic or crystalline symmetries.

The equilibrium between these two terms will result in the emergence of pattern formation. In particular we will show that the minimizers are periodic stripes.

12:30–12:50 The elastica functional as the critical Gamma-limit of the screened Gamow model

Theresa Simon (University of Muenster, Germany)

Abstract. I will consider the large mass limit of a nonlocal isoperimetric problem in two dimensions with screened Coulomb repulsion. In this regime, the competition between perimeter and the repulsion simplifies to leading order by the nonlocal interaction localizing on the boundary of the sets. For an appropriate choice of screening constant, the surface area is exactly cancelled, requiring an analysis of the next order contribution. It turns out that then the nature of the problem changes from length minimization to curvature minimization: I will prove that the Gamma limit is given by (the relaxation of) the elastica functional, i.e., the integral over the squared curvature over the boundary.

14:30–15:15 Nonlocal isoperimetric problems: lamellar pattern, lens cluster, and a new partitioning problem

Lia Bronsard (McMaster University, Canada)

Abstract. We first present a nonlocal isoperimetric problem for three interacting phase domains, related to the Nakazawa-Ohta ternary inhibitory system which describes domain morphologies in a triblock copolymer. We consider global minimizers on the two-dimensional torus, in the droplet regime where some of the species have vanishingly small mass but the interaction strength is correspondingly large. In this limit there is splitting of the masses, and each vanishing component rescales to a minimizer of an isoperimetric problem for clusters in 2D. These results have led to a new type of partitioning problem that I will also describe. These represent work with S. Alama, X. Lu, C. Wang, S. Vriend and M. Novack.

15:30–16:15 An Infinite Double Bubble Theorem

Michael Novack (Carnegie Mellon University, USA)

Abstract. The classical double bubble theorem characterizes the minimizing partitions of $\mathbb{R}^n$ into three chambers, two of which have prescribed finite volume. In this talk we will discuss a variant of the double bubble theorem in which two of the chambers have infinite volume. Such a configuration is an example of a *(1,2)-cluster*, or a partition of $\mathbb{R}^n$ into three chambers, two of which have infinite volume and only one of which has finite volume. For $(1,2)$ clusters, the analogue of the double bubble is the *lens cluster*, and we show that it is minimizing. Furthermore, we show that it is is unique in $\mathbb{R}^n$ for $n \leq 7$, with the same uniqueness holding in $\mathbb{R}^n$ for $n \geq 8$ under a natural growth assumption.

17 –17:45 A max-min property of the ball

Almut Burchard (University of Toronto, Canada)

Abstract. I will present recent joint work with Davide Carazzato and Ihsan Topaloglu on maximizing a functional that involves the minimization of the Wasserstein distance between disjoint sets of equal volume. The functional appears as a repulsive interaction term in some models describing biological membranes. Using a symmetrization-by-reflection technique, we show that balls are the unique maximizers of the functional.

July 24, 2024

11:30–11:50 Diffuse improvement of flatness in codimension two

Alessandro Pigati (University of Pisa, ITALY)

Abstract. The Allen–Cahn energy

$$F_\varepsilon(u) = \int \left[\epsilon |du|^2 + \frac{(1-u^2)^2}{4\epsilon} \right]$$

for real-valued maps u is by now a well-understood way to approximate the area functional for hypersurfaces. Critical points of it converge to minimal hypersurfaces as we send the scaling parameter ε to zero, and the same holds for the gradient flow, which converges to the mean

curvature flow for hypersurfaces. Inspired by this parallel, De Giorgi proposed a conjecture which is analogous to the Bernstein problem for minimal graphs: *given an entire critical point* $u : \mathbb{R}^n \to \mathbb{R}$ *in dimension* $n < 9$*, monotone in* x_n*, is it necessarily a function of just* x_n*, up to rotations?*

Savin solved this conjecture for minimizers, which can be seen to be equivalent to a mild additional assumption. We discuss an analogue in codimension two, for the abelian Yang–Mills–Higgs energy

$$E_\varepsilon(u, \nabla) = \int \left[|\nabla u|^2 + \frac{(1-|u|^2)^2}{4\varepsilon^2} + \varepsilon^2 |F_\nabla|^2 \right],$$

which is known to approximate area in codimension two; here u is a complex-valued map and $\nabla = d - i\alpha$ a connection on the trivial complex line bundle, with curvature F_∇. The result is based on an improvement of flatness in the style of Allard and is partly inspired by an alternative proof of Savin's theorem by Wang. It also uses recent stability results in dimension two by Halavati.

This is joint work with Guido De Philippis (NYU Courant) and Aria Halavati (NYU Courant).

12:00–12:20 Topological spin textures stabilized by long range dipolar interaction in ferromagnetic thin films and their applications

Anne Bernand-Mantel (CEMES-CNRS, FRANCE)

Abstract.

Topological spin textures are potential bit-encoding states for various information technology applications including high-density/high-speed memory and unconventional computing such as neuromorphic, probabilistic and reservoir computing. As a consequence, these systems have been widely studied experimentally and theoretically in recent years. Nevertheless, despite numerous attempts, a satisfactory theoretical description of these objects is still lacking today due to the highly non-trivial character of the magnetostatic interaction that plays a major role in determining the nature of magnetization patterns in ferromagnets. In this talk we will present a pedagogical overview of magnetization patterns existing in the case of ferromagnetic thin films. We will build a phase diagrams to locate magnetic patterns (uniform, stripe, helicoid, bubble, skyrmion...) as a function of the ferromagnetic layer thickness, applied magnetic field and other physical parameters. We will start with simplified "toy" models enabling physical insight of what a ferromagnet is and the origin of hysteresis (Stoner-Wohlfarth, thin wall models...). We will continue with results involving rigorous mathematical analysis [1-4]. This talk will be illustrated with experimental and numerical observations. We will finally give a concrete example of a promising applications using topological spin textures for ultra-fast/ultra-low power physical random number generation with various applications from probabilistic AI to Monte Carlo based intensive calculations in finance.

12:30–12:50 Internal structure of the 2-pi wall

Hans Knuepfer (University of Heidelberg, Germany)

Abstract. The 2 pi wall is a type of transition layer which appears in thin ferromagnetic films. We discuss existence and structure of the 2 pi wall from the variational point of view. This is joint work with A. Capella and C. Muratov

14:30–14:50 On the stability of the ball for attractive-repulsive energies

Marco Bonacini (University of Trento, ITALY)

Abstract.

In this talk I will consider a class of attractive-repulsive energies, given by the sum of two nonlocal interactions with power-law kernels, defined over sets with fixed measure. It has recently been proved by R. Frank and E. Lieb that the ball is the unique (up to translation) global minimizer for sufficiently large mass. After a review of the literature on this minimization problem, I will focus on the issue of the stability of the ball, in the sense of the positivity of the second variation of the energy with respect to smooth perturbations of the boundary of the ball. I will present a characterization of the range of masses for which the second variation is positive definite (large masses) or negative definite (small masses). Moreover, I will discuss the

connection between the stability of the ball and its local/global minimality. This talk is based on a work in collaboration with Riccardo Cristoferi and Ihsan Topaloglu.

15:00–15:45 Minimizers of Energies with Repulsive-Attractive Power Law Interactions

Ryan Matzke (Vanderbilt University, USA)

Abstract.

We will discuss the minimization of energies with certain repulsive-attractive potentials

$$I_{\alpha,\beta}(\mu) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \Big(\frac{\|x-y\|^{\alpha}}{\alpha} - \frac{\|x-y\|^{\beta}}{\beta} \Big) d\mu(x) d\mu(y).$$

The equilibrium measure of such an energy arises naturally as a steady state of an aggregation model where particles repel one another at short range, but are attracted to each other when far apart. We are particularly interesting in how the choices of β and α, i.e. the strength of repulsion at short range and strength of attraction at long range, respectively, affect the dimension of the support of the minimizing measure, and will provide new results for when the support of the equilibrium is a sphere or a ball.

17:00–17:45 Generative Adversarial Networks: A study of the Dynamics

Matias Delgadino (UT Austin, USA)

Abstract.

Generative Adversarial Networks (GANs) was one of the first Machine Learning algorithms to be able to generate remarkably realistic synthetic images. In this presentation, we delve into the mechanics of the GAN algorithm and its profound relationship with optimal transport theory. Through a detailed exploration, we illuminate how GAN approximates a system of PDE, particularly evident in shallow network architectures. Furthermore, we investigate the phenomenon of mode collapse, a well-known pathological behavior in GANs, and elucidate its connection to the underlying PDE framework through an illustrative example.

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