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Configurations in projective spaces and related research in commutative algebra and algebraic geometry
Special Session A8

Sankhaneel Bisui
Arizona State University, USA

Giuseppe Favacchio
University of Palermo, ITALY

Elena Guardo
University of Catania, ITALY

Abu Thomas
Georgia State University, USA

The study of subvariety arrangements in projective spaces is a well-established and highly fruitful area of research. This field has produced numerous profound discoveries that have had a significant impact across various branches of mathematics.

In recent years, there has been a growing interest in studying configurations of points, lines, conics, and hyperplanes. These configurations are not only intriguing in their own right but also serve as powerful tools for uncovering unexpected properties within Commutative Algebra and Algebraic Geometry.

Representative examples of the importance of subvariety arrangements in Commutative Algebra and Algebraic Geometry include the use of grids, root systems, and Kochen-Specker Sets (including the Penrose dodecahedron) to analyze properties related to projections (such as Geproci sets, Weddle varieties). Several configurations give examples of unexpected curves, cones, and hypersurfaces. Contact star configurations were used to give a geometrical interpretation of some variety defined by Hadamard products. Additionally, arrangements like the Fermat arrangement, Star and Steiner configurations, and various others have played a crucial role in investigating containment problems, with a focus on the Waldschmidt constant, resurgence, and the conjectures of Demailly and Chudnovsky. Such research is often connected with the study of monomial ideals and, in particular, ideals defined over graphs and hypergraphs.

Our central aim is to stimulate open dialogue among researchers, promoting the sharing of progress and novel ideas. Through this, we want to encourage experts from different mathematical areas to collaborate on problems collectively. We hope to establish an enduring connection among researchers, fostering a spirit of collaboration that persists even after the session ends.

Schedule and Abstracts

July 23, 2024

11:00–11:40 Adding generators to monomial ideals.
Sara Faridi (University of British Columbia, CANADA)

Abstract. In this talk we will show how the concept of an "elementary collapse" from discrete homotopy theory can be used to enlarge a given monomial ideal, while preserving most of its homological properties.

11:50–12:10 "To infinity... and beyond!" with initial degrees of configurations of points.
Roberta Di Gennaro (Università degli Studi di Napoli-Federico II, ITALY)

Abstract. In [1] and [2] the authors study the Waldschmidt constant

$$\widehat{\alpha}(I_{\mathbb{X}}) = \lim_{t\to\infty} \frac{\alpha({I_{\mathbb{X}}}^{(t)})}{t}$$

of the ideal $I_{\mathbb{X}}$ defining k-configurations and standard k-configurations (which are special k-configurations of points whose coordinates are integer values). A k-configuration of points in $\mathbb{P}^2$ is a finite set $\mathbb{X}$ of points in $\mathbb{P}^2$ decomposable as $\mathbb{X} = \bigcup_{i=1}^{s} \mathbb{X}_i$ supported on suitable lines L_i $(1 < i \leqslant s)$ such that L_i contains $\mathbb{X}_i$ for each $i = 1, \ldots, s$ but does not contain any point of $\mathbb{X}_j$ for all $j < i$.

Here, we present some recent advances on the Waldschmidt constant of configurations of points in $\mathbb{P}^2$ supported on lines but that are not standard k-configurations, called partial intersections. It is a work currently in progress based on a joint project arised during the WICA II workshop (CIRM - Trento).

References

[1] M.V. Catalisano, E. Guardo, Y.S. Shin. The Waldschmidt constant of special k-configurations in $\mathbb{P}^n$. J. Pure Appl. Algebra, **224**(10):106341, (2020).

[2] M.V. Catalisano, G. Favacchio, E. Guardo, Y.S. Shin. The Waldschmidt constant of a standard k-configurations in $\mathbb{P}^2$. arXiv:2307.06014.

12:20–12:40 Lefschetz properties of squarefree monomial ideals via Rees algebras.
Thiago Holleben (Dalhousie University, CANADA)

Abstract. The theory of Rees algebras of monomial ideals has been extensively studied, and as a consequence, many (sometimes partial) equivalences between algebraic properties of monomial ideals, and combinatorial properties of simplicial complexes and hypergraphs are known. In this talk, we will see how this theory can be used to find interesting examples in the theory of Lefschetz properties. We will also explore the consequences of known results from Lefschetz properties to the Rees algebras of squarefree monomial ideals. Applications include a connection between symbolic powers and f-vectors of simplicial complexes, and the positivity of mixed multiplicities of some families of squarefree monomial ideals.

References

[1] T. Holleben, *Lefschetz properties of squarefree monomial ideals via Rees algebras*, arxiv:2404.12471, 2024.

[2] T. Holleben, *The weak Lefschetz property and mixed multiplicities of monomial ideals*, arxiv:2306.13274, 2023.

14:30–15:10 Generalized Weddle loci, tensors, and the geometry of projections.
Luca Chiantini (University of Siena, ITALY)

Abstract. I will introduce the notion of generalized Weddle loci (see [3]), which are important in the study of configurations of points in projective spaces and in computer vision, and I will discuss some recent application of the loci to the description of special configurations, projections, and spaces of tensors. I will mainly focus on a work in progress related the use of generalized Weddle loci for determining the structure of geproci sets (see [1]), the rank of linear systems of quadrics (see [2]), and the structure of spaces of partially symmetric tensors.

References

[1] Chiantini L., Farnik Ł., Favacchio G., Harbourne B., Migliore J., Szemberg T., Szpond J., *Finite sets in projective spaces and their projections*, preprint arXiv2209.04820.

[2] Itin Y., Reches S., *Decomposition of third-order constitutive tensors*, Math. Mech. Solids. 27(402) (2021).

[3] Weddle T., *On the theorems in space analogous to those of Pascal and Brianchon in a plane. Part II*, Cambridge and Dublin Mathematical Journal, 5 (1850), pp. 58–69.

15:20–15:40 Towards the classification of geproci sets.
Łucja Farnik (University of the National Education Commission, Krakow, POLAND)

Abstract. A geproci set of points is a set whose general projection to a hyperplane is a complete intersection. We give examples of geproci sets in characteristic zero and present the so-called

standard construction of nongrid geproci sets. Moreover we discuss recent results in the classification of geproci sets on skew lines having two transversals.

The talk is based on a joint project with Luca Chiantini, Pietro De Poi, Giuseppe Favacchio, Brian Harbourne, Giovanna Ilardi, Juan Migliore, Tomasz Szemberg and Justyna Szpond.

References

[1] L. Chiantini, P. De Poi, Ł. Farnik, G. Favacchio, B. Harbourne, G. Ilardi, J. Migliore, T. Szemberg, J. Szpond, *Geproci sets on skew lines in* $\mathbb{P}^3$ *with two transversals*, arXiv:2312.04644.

[2] L. Chiantini, Ł. Farnik, G. Favacchio, B. Harbourne, J. Migliore, T. Szemberg, J. Szpond, *Configurations of points in projective space and their projections*, arXiv:2209.04820.

[3] L. Chiantini, Ł. Farnik, G. Favacchio, B. Harbourne, J. Migliore, T. Szemberg, J. Szpond, *On the classification of certain geproci sets*, arXiv:2303.16263, to appear in the proceedings of the 2022 Workshop in Cortona: "Lefschetz Properties: Current and New Directions".

[4] L. Chiantini, Ł. Farnik, G. Favacchio, B. Harbourne, J. Migliore, T. Szemberg, J. Szpond, *Geproci sets and the combinatorics of skew lines in* $\mathbb{P}^3$, arXiv:2308.00761.

[5] L. Chiantini, J. Migliore, *Sets of points which project to complete intersections, and unexpected cones*, Trans. Amer. Math. Soc., 374(4) (2021), 2581–2607. With an appendix by A. Bernardi, L. Chiantini, G. Denham, G. Favacchio, B. Harbourne, J. Migliore, T. Szemberg and J. Szpond.

15:50–16:10 The geproci property in positive characteristic.
Jake Kettinger (Colorado State University, USA)

Abstract. The geproci property is a recent development in the world of geometry. We call a set of points $Z \subseteq \mathbb{P}^3_k$ an (a, b)-geproci set (for GEneral PROjection is a Complete Intersection) if its projection from a general point P to a plane is a complete intersection of curves of degrees $a \leq b$. Nondegenerate examples known as grids have been known since 2011. Nondegenerate nongrids were first described in 2018, working in characteristic 0. Almost all of these new examples are of a special kind called half grids.

In this paper, based partly on the author's thesis, we use a feature of geometry in positive characteristic to give new methods of producing geproci half grids and non-half grids.

Theorem 1. Let $\mathbb{F}_q$ be the field of size q, where q is some power of a prime. Then $Z = \mathbb{P}^3_{\mathbb{F}_q} \subseteq \mathbb{P}^3_{\overline{\mathbb{F}}_q}$ is a $(q + 1, q^2 + 1)$-geproci half grid.

Theorem 2. The complement $\widetilde{Z} \subseteq \mathbb{P}^3_{\mathbb{F}_q}$ of a maximal partial spread of deficiency d is a nontrivial $\{q + 1, d\}$-geproci set. Furthermore, when $d > q + 1$, $\widetilde{Z}$ is also not a half grid.

17:00–17:20 Construction of free and nearly free curves.
Giovanna Ilardi (Università degli studi di Napoli-Federico II, ITALY)

Abstract. We construct new free or nearly free curves starting from a curve C by adding inflectional tangents, or lines passing through the singularities of C, or lines in the tangent cone of some singularities of the curve. We give new results and construct many examples (joint work with A. Dimca, P. Pokora and G. Sticlaru).

Then we construct new families of free and nearly free curves starting from a plane cubic curve C and adding some of its hyperosculating conics, that is conics with the property that the intersection index of such a conic with a given cubic is exactly 6.

We get new insight into the geometry of the 27 hyperosculating conics of the Fermat cubic curve (joint work with A.Dimca, G. Malara, P. Pokora).

17:30–17:50 Gotzmann's persistence theorem for products of projective spaces.
Patience Ablett (University of Warwick, UK)

Abstract. Gotzmann's regularity and persistence theorems provide tools which allow us to find explicit equations for the Hilbert scheme $\mathrm{Hilb}_P(\mathbb{P}^n)$. A natural next step is to generalise these results to the multigraded Hilbert scheme $\mathrm{Hilb}_P(X)$ of a smooth projective toric variety X. In 2003 Maclagan and Smith ([1],[2]) generalise Gotzmann's regularity theorem to this case. We

present new persistence type results for the product of two projective spaces, and time permitting discuss how these may be applied to a more general smooth projective toric variety.

References

[1] D. Maclagan, G. Smith, *Multigraded Castelnuovo-Mumford regularity*, J. Reine Angew. Math, 571 (2004), 179–212

[2] D. Maclagan, G.Smith, *Uniform bounds on multigraded regularity* J. Algebraic Geom, 14 (2005), 137–164

July 24, 2024

11:30–12:10 Configurations with triple points.
Halszka Tutaj-Gasińska (Jagiellonian University, POLAND)

Abstract. The talk concerns configurations with the maximal number of triple points. We show a construction of an infinite series of line arrangements (in characteristic two) with only triple intersection points. We will also present some results on the existence and non-existence of line arrangements (in various characteristics) with up to 19 lines maximizing the number of triple points.

12:20–13:00 Zero-dimensional schemes via Hadamard products.
Cristiano Bocci (University of Siena, ITALY)

Abstract. In this talk I will show how to use Hadamard products of projective varieties to construct set of points which are star configurations, fat grids or Gorenstein with given $h-$vector.

References

[1] I.B. Jafarloo, C. Bocci, E. Guardo, G. Malara, *Hadamard Products of Symbolic Powers and Hadamard Fat Grids*. Mediterranean Journal of Mathematics, **20**(162) (2023).

[2] C. Bocci, C. Capresi, D. Carrucoli, *Gorenstein points in $\mathbb{P}^3$ via Hadamard products of projective varieties*, Coll. Mathematica (2022).

[3] C. Bocci, E. Carlini, *Hadamard products of projective varieties*, Frontiers in Mathematics, Springer, 2024.

14:30–15:10 On rational complete intersections.
Francesco Russo (Università degli studi di Catania, ITALY)

Abstract. The known results about the rationality vs irrationality of Fano complete intersections $X^n \subset \mathbb{P}^{n+c}$ of dimension $n = 3, 4, 5$ and fixed type $(d_1, \ldots, d_c)$ suggest an uniform approach to treat several open cases: index one; index two; quartic fourfolds and fivefolds; etc). From one hand one would like to decide the rationality/irrationality of every element in the numerous cases where the stable irrationality of the very general element is known (e.g. quartic fourfolds and fivefolds, quintic fivefolds, etc); from the other hand one hopes to put some further light on several longstanding conjectures (e.g. the irrationality of the very general cubic fourfold).

15:20–15:40 Complete intersections on Veronese surfaces.
Stefano Canino (Politecnico di Torino, ITALY)

Abstract. In "Commentationes Geometricae" Euler asked when a sets of points in the plane is the intersection of two curves, that is, using the modern terminology, when a set of points in the plane is a complete intersection. In the same period, Cramer asked similar questions so that this type of questions is presently known as the Cramer-Euler problem. In this talk, we consider a generalization of the Cramer-Euler problem: characterize the possible complete intersections lying on a Veronese surface, and more generally on a Veronese variety. The main result describes all possible reduced complete intersections on Veronese surfaces. More precisely, we prove the following theorem:

Theorem. If $\mathbb{X} \subseteq V_{2,d} \subseteq \mathbb{P}^{N_{2,d}}$ is a reduced complete intersection of type $(a_1, \ldots, a_r)$, with $a_1 \leq \cdots \leq a_r$, then one of the following holds:

(1) $(d, r, (a_1, a_2, \ldots, a_r)) = (2, 4, (1, 1, 1, 2))$, that is, $\mathbb{X}$ is a conic lying on $V_{2,2}$;

(2) $(d,r,(a_1,a_2,\ldots,a_r)) = (2,5,(1,1,1,2,a_5))$, any $a_5 \in \mathbb{N}$, that is, $\mathbb{X}$ is a set of $2a_5$ complete intersection points of a conic lying on $V_{2,2}$ and a hypersurface of degree a_5;
(3) $(d,r,(a_1,a_2,\ldots,a_r)) = (d,N_{2,d},(1,1,\ldots,1))$ for any $d \geq 2$, that is, $\mathbb{X}$ is a reduced point;
(4) $(d,r,(a_1,a_2,\ldots,a_r)) = (d,N_{2,d},(1,1,\ldots,1,2))$ for any $d \geq 2$, that is, $\mathbb{X}$ is a set of two reduced points.

We formulate a conjecture for the general case of complete intersection subvarieties of any dimension, and we prove it in the case of the quadratic Veronese threefold. Our main tool is an effective characterization of all possible Hilbert functions of reduced subvarieties of Veronese surfaces.

Conjecture. Let $\mathbb{X} \subseteq V_{n,d} \subseteq \mathbb{P}^{N_{n,d}}$ be a reduced subvariety with $d > 1$. Then $\mathbb{X}$ is a complete intersection of type $(a_1,\ldots,a_r)$, with $a_1 \leq \cdots \leq a_r$ if and only if

- $r = N_{n,d}$, $a_1 = \ldots = a_{N_{n,d}} = 1$, any n,d, that is $\mathbb{X}$ is a reduced point;
- $r = N_{n,d}$, $a_1 = \ldots = a_{N_{n,d}-1} = 1$, $a_{N_{n,d}} = 2$, any n,d, that is $\mathbb{X}$ is a set of two reduced points;
- $r = N_{n,d}$, $a_1 = \ldots = a_{N_{n,d}-2} = 1$, $a_{N_{n,d}-1} = 2$, $a_{N_{n,d}} = b$, any n, $d = 2$, any $a \geq 2$, that is $\mathbb{X} = \mathcal{C} \cap H_b$ for $\mathcal{C} \subseteq V_{n,2}$ a conic and H_b a degree b hypersurface;
- $r = N_{n,d} - 1$, $a_1 = \ldots = a_{N_{n,d}-2} = 1, a_{N_{n,d}-1} = 2$, $d = 2$, any n, that is $\mathbb{X}$ is a conic.

15:50–16:10 Toric Ideals of Graphs and the Fundamental Group.
Graham Keiper (Università degli studi di Catania, ITALY)
Abstract. This talk will discuss a surprising connection between combinatorial commutative algebra and algebraic topology. Specifically, for a connected biparite finite simple graph G, we will look at the relationship between the toric ideal associated with G, I_G, and the fundamental group of G, $\pi_1(G)$. We will examine how properties of the fundamental group $\pi_1(G)$ allow us to obtain important algebraic information about the toric ideal I_G.

17:00–17:40 Simplices osculating to rational normal curves.
Enrico Carlini (Politecnico di Torino, ITALY)
Abstract. In this talk we discuss one of the many variations of the following question: does there exist a rational normal curve in $\mathbb{P}^d$ passing through n given points? For $n = d+3$ points in general linear position an affirmative answer is given by the well known Castelnuovo's Lemma. While, for any collection of $n \geq d+4$ points, the paper [1] provides explicit equations that always give an answer to the question.

We will consider special sets of points, namely the vertices of two simplices whose faces osculate a given rational normal curve. For example, one can consider $d = 2, n = 6$ that is the vertices of two triangles osculating to a conic. And also, $d = 3, n = 8$, that is two tetrahedrons osculating to a twisted cubic curve. This particular instance was considered and answered by von Staudt in 1856, by Hurwitz in 1882, and by White in 1921. In particular, White also claimed to have an argument to prove the general case, but a proof was not provided.

Our main result in [2] is the following:

Theorem. Let $d \geq 2$ and let $\mathcal{C} \subseteq \mathbb{P}^d$ be a rational normal curve. Consider distinct points $P_i, P'_i \in \mathcal{C}$ for $1 \leq i \leq d+1$ and let π_i, respectively π'_i, be the osculating hyperplanes to $\mathcal{C}$ in P_i, respectively P'_i. Define points

$$R_i = \bigcap_{j \neq i} \pi_j \text{ and } R'_i = \bigcap_{j \neq i} \pi'_j.$$

Then, the points $R_1,\ldots,R_{d+1},R'_1,\ldots,R'_{d+1}$ lie on a rational normal curve.

References

[1] Caminata, Alessio and Giansiracusa, Noah and Moon, Han-Bom and Schaffler, Luca, *Equations for point configurations to lie on a rational normal curve*,Adv. Math., 340 (2018), 653–683.

[2] Caminata, Alessio and Carlini, Enrico and Schaffler, Luca, *Simplices osculating to rational normal curves* preprint https://arxiv.org/abs/2404.03922.

17:50–18:10 On the codimension of permanental varieties.
Emanuele Ventura (Politecnico di Torino, ITALY)
Abstract. We study *permanental varieties*, i.e. varieties defined by the vanishing of permanents of fixed size of a generic matrix. Permanents and their varieties play an important, and sometimes poorly understood, role in combinatorics. However, there are essentially no geometric results about them in the literature, in very sharp contrast to the well-behaved and ubiquitous case of determinants and minors. Motivated by the study of the singular locus of the permanental hypersurface, we focus on the codimension of these varieties. We introduce a $\mathbb{C}^*$-action on matrices and prove a number of results. In particular, we improve a lower bound on the codimension of the aforementioned singular locus established by von zur Gathen in 1987. This is joint work with Boralevi, Carlini and Michałek.

E–mail: `giuseppe.favacchio@unipa.it`.