Geometric Analysis of Several Complex Variables and PDEs
Special Session B5

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The section, scheduled on July 25-26, will feature talks by leading experts on recent developments in the analysis and geometry of CR manifolds and related partial differential equations. It is expected that the session will promote collaborations between researchers in the U.S. and Italy.

Schedule and Abstracts

July 25, 2024

11:30–11:55 Dynamics of Fuchsian meromorphic connections with real periods
Marco Abate (University of Pisa, ITALY)

Abstract. A very interesting class of examples of holomorphic maps tangent to the identity at a fixed point in several complex variables is given by the time-1 maps of homogeneous vector fields. In [1] it has been shown that the study of the dynamics of these maps can be reduced to the study of the dynamics of the geodesic field of meromorphic connections on Riemann surfaces. In this talk we shall describe some recent results, obtained in collaboration with Karim Rakhimov, on the dynamics of the geodesic field for Fuchsian meromorphic connections having real periods. The main tools used are: a generalization to general Fuchsian meromorphic connections of a classical formula proved by Teichmüller for quadratic differentials; and the relationship between Fuchsian meromorphic connections with real periods and singular flat Hermitian metrics. In particular, we obtain a description of the possible ω-limit sets of simple geodesics that extends and makes more precise results known for the particular case of Riemann surfaces endowed with a meromorphic k-differential.

12:00–12:25 Global hypoellipticity and solvability for evolution operators in time-periodic Gelfand-Shilov spaces
Marco Cappiello (University of Turin, ITALY)

Abstract. We consider a class of evolution operators with complex-valued coefficients depending both on time and space variables $(t, x) \in \mathbb{T} \times \mathbb{R}^n$, where $\mathbb{T}$ is the one-dimensional torus. We prove necessary and sufficient conditions for global hypoellipticity and solvability in spaces of time-periodic Gelfand-Shilov functions. The argument of the proof is based on a characterization of these spaces in terms of the eigenfunction expansions given by a fixed self-adjoint, globally elliptic differential operator on $\mathbb{R}^n$, cf. [3]. The results presented are obtained in collaboration with Fernando de Ávila Silva (Federal University of Paraná, Curitiba, Brazil), cf. [1,2].

References

[1] F. de Ávila Silva, M. Cappiello, *Time-periodic Gelfand-Shilov spaces and global hypoellipticity on* $\mathbb{T} \times \mathbb{R}^n$, J. Funct. Anal. n. 9, 109418 (2022).

[2] F. de Ávila Silva, M. Cappiello, *Globally solvable time-periodic evolution equations in Gelfand-Shilov classes*, Preprint 2024, https://arxiv.org/abs/2402.10006

[3] T. Gramchev, S. Pilipović, L. Rodino, *Eigenfunction expansions in* $\mathbb{R}^n$, Proc. Amer. Math. Soc. **139** n. 12, 4361–4368 (2011).

12:30–13:00 On the local behavior of chains on strongly pseudoconvex hypersurfaces in $\mathbb{C}^3$
Giuseppe Della Sala (American University of Beirut, LEBANON)

Abstract. We study the behavior of Chern-Moser chains that wind around a small enough neighborhood of a point $p \in M$, where p is a smooth, strongly pseudoconvex hypersurface of $\mathbb{C}^3$. We prove that, if for all $p \in M$ these chains are topologically "tame" in the sense that in appropriate coordinates and almost horizontal initial conditions their images are small perturbations of circles contained in $T^c_p(M)$, then M is locally spherical. This is joint work with Nisrine Bakaev, Seok Ban, Hassan El Bouz, Ahmad Hussein and Jean Moussa.

14:30–14:55 Bergman logarithmically flat and obstruction flat CR manifolds
Peter Ebenfelt (University of California at San Diego, USA)

Abstract. Let $\Omega \subset \mathbb{C}^n$ be a smoothly bounded, strictly pseudoconvex domain. The boundary $\partial\Omega$ is said to be *Bergman logarithmically flat* if the log singularity in Fefferman's asymptotic expansion of the Bergman kernel vanishes (to infinite order). It is called *obstruction flat* if the log singularity (the obstruction function) of the Cheng-Yau log-potential of the complete Kähler-Einstein metric in Ω vanishes. The Ramadanov Conjecture asserts that if $\partial\Omega$ is Bergman logarithmically flat, then it is spherical. There is a similar conjecture for obstruction flat boundaries. Both conjectures, suitably reformulated, fail for domains in more general complex manifolds in higher dimension ($n \geq 3$), but the situation is still unclear for domains in $\mathbb{C}^n$ (for $n \geq 3$). In this talk, we shall present recent work and open questions concerning these conjectures and the general structure of Bergman logarithmically flat and obstruction flat CR manifolds.

15:00–15:25 Equivalence between validity of the p-Poincaré inequality and finiteness of the strict p-capacitary inradius
Anne-Katrin Gallagher (Gallagher Tool & Instrument, USA)

Abstract. We will talk about some aspects of the equivalence of the validity of the p-Poincaré inequality on an open set Ω in $\mathbb{R}^n$, i.e.,

$$\|f\|_{p,\Omega} \leq C\|\nabla f\|_{p,\Omega} \quad \forall\, f \in \mathcal{C}^\infty_c(\Omega)$$

for some $C > 0$, and the finiteness of the p-capacitary inradius, $\rho_p(\Omega)$, of Ω defined by

$$\rho_p(\Omega) = \sup\{r > 0 \ : \ \forall\, \epsilon > 0 \exists\, x \in \mathbb{R}^n \text{ such that } C_p(\overline{\mathbb{B}_r(x)} \cap \Omega^c) < \epsilon\}$$

Here, $C_p(E)$ denotes the Sobolev p-capacity of E for $E \subset \mathbb{R}^n$.

15:30–15:55 A structure theorem for neighborhoods of compact complex manifolds
Xianghong Gong (University of Wisconsin-Madison, USA)
Laurent Stolovitch (Université Côte d'Azur, France)

Abstract. We show that the set of holomorphic equivalence classes of holomorphic neighborhoods M of a compact complex manifold C is finite-dimensional, provided $(TM)|_C$ is fixed and the normal bundle of C in M is either *weakly negative* or 2-*positive*.

Our main result is the following.

Theorem 1. *Let C be a compact complex manifold. Assume that N_C is either weakly negative or 2-positive. There is an injective mapping from the set of holomorphic equivalence classes of holomorphic neighborhoods of C into the finite-dimensional space*

$$\mathcal{H}^1(T_C M) := \bigoplus_{\ell \geq 2} H^1(C, T_C M \otimes S^\ell N_C^*),$$

where $S^\ell N_C^*$ is the ℓ-th symmetric power of the dual bundle N_C^* of N_C.

16:00–16:25 Semi-classical asymptotics of partial Bergman kernels on R-symmetric complex manifolds with boundary

Xiaoshan Li (Wuhan University, CHINA)
Chin-Yu Hsiao (Institute of Mathematics, Academia Sinica, TAIWAN)
George Marinescu (University of Cologne, GERMANY)

Abstract. Let M be a relatively compact connected open subset with smooth connected boundary of a complex manifold M'. Let (L, h^L) be a positive line bundle over M'. Suppose that M' admits a holomorphic $\mathbb{R}$-action which preserves the boundary of M and lifts to L. In this talk, we will show an asymptotic expansion of a partial Bergman kernel associated to a package of Fourier modes of high frequency with respect to the $\mathbb{R}$-action in the high powers of L. As an application, we establish an $\mathbb{R}$-equivariant analogue of Fefferman's and Bell-Ligocka's result about smooth extension up to the boundary of biholomorphic maps between weakly pseudoconvex domains in $\mathbb{C}^n$. Another application concerns the embedding of pseudoconcave manifolds.

17:00–17:25 Bumping and sup norm estimates for $\bar{\partial}$ on smooth pseudoconvex domains in $\mathbb{C}^n$

Andreea Nicoara (Trinity College Dublin, IRELAND)

Abstract. The $\bar{\partial}$-Neumann problem was solved in the 60's for pseudoconvex domains, but it took till the 80's for Sobolev estimates to be proven for smooth pseudoconvex domains of finite q-type. Sup norm and Hölder estimates are even harder to obtain. Charlie Fefferman and Joe Kohn handled the problem in $\mathbb{C}^2$ in 1988, and then it took until 2022 for Dusty Grundmeier, Lars Simon, and Berit Stensønes to publish a solution in dimension three for real-analytic pseudoconvex domains of finite D'Angelo 1-type. Their solution involves bumping, namely constructing a bigger domain and a special support function that allows for the uniformity of the estimates to be obtained. I will discuss work in progress with Nicholas Aidoo, John Erik Fornæss, and Berit Stensønes (posthumously) in order to provide the solution in dimension n for smooth pseudoconvex domains of D'Angelo finite 1-type. Using stratifications and a certain amount of real algebraic geometry, we have constructed the bumping, and we are currently working through the estimates.

17:30–17:55 Quasi-finite typeness and 1-regular types on algebraic CR manifolds: global boundedness I

Bernhard Lamel (Texas A&M University at Qatar, Qatar)

Abstract. We present recent results about finite jet determination of CR maps of positive codimension from real-analytic CR manifolds into Nash manifolds (or sets) in complex space. One instance of such results is the unique jet determination of germs of CR maps from minimal real-analytic CR submanifolds in $\mathbb{C}^N$ into Nash subsets of $\mathbb{C}^N$ of D'Angelo finite type, for arbitrary $N, N \geq 2$. One the ingredients in the proof relies on the global boundedness of the quasi-finite type for Nash maps discussed in the earlier special session "Several Complex Variables: Theory and Applications". This is joint work by B. Lamel, N. Mir and G. Rond.

References

[1] B. Lamel, N. Mir, G. Rond: Unique jet determination of CR maps into Nash sets, *Adv. Math.* 432 (2023), 109271. `https://doi.org/10.1016/j.aim.2023.109271`

18:00–18:30 Finite jet determination of CR maps in positive codimension II

Nordine Mir (Texas A&M University at Qatar, Qatar)

Abstract. We present recent results about finite jet determination of CR maps of positive codimension from real-analytic CR manifolds into Nash manifolds (or sets) in complex space. One instance of such results is the unique jet determination of germs of CR maps from minimal real-analytic CR submanifolds in $\mathbb{C}^N$ into Nash subsets of $\mathbb{C}^N$ of D'Angelo finite type, for arbitrary

$N, N \geq 2$. One the ingredients in the proof relies on the global boundedness of the quasi-finite type for Nash maps discussed in the earlier special session "Several Complex Variables: Theory and Applications". This is joint work by B. Lamel, N. Mir and G. Rond.

References

[1] B. Lamel, N. Mir, G. Rond: Unique jet determination of CR maps into Nash sets, *Adv. Math.* 432 (2023), 109271. https://doi.org/10.1016/j.aim.2023.109271

July 26, 2024

11:30–11:55 Optimal data spaces for boundary value problems for the d-bar operator on Lipschitz planar domains
Loredana Lanzani (University of Bologna, ITALY)

Abstract. We study the $\overline{\partial}$-equation subject to various boundary value conditions on bounded simply connected Lipschitz domains $D \Subset \mathbb{C}$: for the Dirichlet problem with datum in $L^p(bD, \sigma)$, this is simply a restatement of the fact that members of the holomorphic Hardy spaces are uniquely and completely determined by their boundary values. Here we identify the maximal data spaces and obtain estimates in the maximal p-range for the Dirichlet, Regularity-for-Dirichlet, Neumann, and Robin boundary conditions for $\overline{\partial}$.

This is joint work with W. E. Gryc (Muhlenberg College), J. Xiong (U. of Colorado) and Y. Zhang (Purdue U. - Fort Wayne).

12:00–12:25 Some advances in analytic hypoellipticity
Marco Mughetti (University of Bologna, ITALY)

Abstract. In this talk we discuss the problem of the real analytic regularity of the solutions to sums of squares of vector fields. While the problem of the C^∞ hypoellipticity has been settled from the very beginning by Hörmander, the problem of the analytic hypoellipticity is still open and seems much more involved.
Treves conjecture states that a “sum of squares"-type operator is analytic hypoelliptic if and only if all the Poisson strata of its characteristic set are symplectic. Although this conjecture does not hold in dimension 4 or greater, some model examples would suggest that the analytic regularity still depends on a suitable stratification of the characteristic variety of the operator.
In dimension 3 we also think that the conjecture does not hold, while we think that in dimension 2 it should be true. As a consequence the low dimensional cases seem to offer some perspective into the problem.

14:30–14:55 On CMC-immersions of surfaces into Hyperbolic 3-manifolds
Gabriella Tarantello (University of Roma "Tor Vergata", ITALY)

Abstract. I discuss a parametrization of the moduli space of Constant Mean Curvature (CMC) c-immersions of a closed surface S (orientable and of genus at least 2) into hyperbolic 3-manifolds, by elements of the tangent bundle of the Teichmueller space of S, provided the (prescribed) mean curvature c satisfies $|c| < 1$.

In addition I shall discuss the asymptotic behaviour, as $|c| \to 1$, of those (CMC)-c immersions, and establish a convergence result in terms of the Kodaira map. For example, in case of genus 2, it is possible to catch at the limit (regular) (CMC)-1 immersions, provided we avoid (in a suitable sense) the image by the Kodaira map of the six Weierstrass points of the given Riemann surface. If time permits, I shall mention further progress for higher genus obtained in collaboration with S. Trapani.

15:00–15:25 Bergman metrics of constant holomorphic sectional curvature
John N. Treuer (University of California at San Diego, USA)

Abstract. In 2023, Huang and Li considered complex manifolds admitting a Bergman metric of constant holomorphic sectional curvature. Building on their work, in this talk we show no complex manifold whose Bergman space is base-point free, separates directions and separates points can have a Bergman metric with identically zero holomorphic sectional curvature.

References

[1] X. Huang and S.-Y. Li, *Bergman metrics as pull-backs of the Fubini-Study metric*, https://arxiv.org/abs/2302.13456, (2023).

15:30–15:55 A new Poincaré type rigidity phenomenon with applications
Ming Xiao (University of California at San Diego, USA)

Abstract. In this talk, we discuss a new Poincaré type phenomenon. More precisely, we will present an optimal rigidity theorem for local CR mappings between circle bundles that are defined in a canonical way over (possibly reducible) bounded symmetric domains. We prove such a local CR map, if nonconstant, must extend to a rational biholomorphism between the corresponding disk bundles. We will also talk about some applications of the theorem.

16:00–16:25 Global regularity in the $\bar{\partial}$-Neumann problem and finite type conditions
Dmitri Zaitsev (Trinity College Dublin, IRELAND)

Abstract. The celebrated work of Catlin on global regularity of the $\bar{\partial}$-Neumann operator for pseudoconvex domains of finite type links local algebraic- and analytic geometric invariants through potential theory with estimates for $\bar{\partial}$-equation. Yet despite their importance, there seems to be a major lack of understanding of Catlin's techniques, resulting in a notable absence of an alternative proof, exposition or simplification.

The goal of my talk will be to present an alternative proof based on a new notion of a "tower multi-type". The finiteness of the tower multi-type is an intrinsic geometric condition that is more general than the finiteness of the regular type, which in turn is more general than the finite type. Under that condition, we obtain a generalized stratification of the boundary into countably many level sets of the tower multi-type, each covered locally by strongly pseudoconvex submanifolds of the boundary. The existence of such stratification implies Catlin's potential-theoretic "Property (P)", which, in turn, is known to imply global regularity via compactness estimate. Notable applications of global regularity include Condition R by Bell and Ligocka and its applications to boundary smoothness of proper holomorphic maps generalizing a celebrated theorem by Fefferman.

17:00–17:25 CR transversality of holomorphic maps between real hypersurfaces
Weixia Zhu (University of Vienna, AUSTRIA)

Abstract. In this talk, I will discuss the history and established results concerning the CR transversality problem, and share our recent progress on real hypersurfaces when the target manifold is a hyperquadric. Specifically, we consider holomorphic maps F from Levi non-degenerate real hypersurfaces $M_\ell \subset \mathbb{C}^n$ to a hyperquadric $\mathbb{H}_\ell^N$ with the same signature ℓ and $N - n < n - 1$. We show that F is either CR transversal to $\mathbb{H}_\ell^N$ or maps a neighborhood of M_ℓ in $\mathbb{C}^n$ into $\mathbb{H}_\ell^N$. This is a joint work with Xiaojun Huang.

17:30–17:55 On the local Gevrey regularity of local Gevrey vectors of Hörmander's operators of degenerate parabolic type
Makhlouf Derridj (Université de Haute Normandie, FRANCE)

Abstract. My aim in this talk is to report on a common work with Gregorio Chinni about the local Gevrey regularity of local Gevrey vectors of the following operators of second order introduced by L. Hörmander in his famous paper in 1967 ("*Hypoelliptic second order differential equations*", Acta Math. **119**, 147-171):

$$P(x, D) = \sum_{j=1}^{m} (X_j(x, D))^2 + X_0(x, D) + c(x),$$

where $X_j(x, D)$, $j = 0, 1, \ldots, m$, are vector fields with real-valued smooth coefficients on an open set Ω in $\mathbb{R}^n$.
The *Hörmander's condition* for hypoellipticity of P in Ω, may be written as follows :
Define the *weight* of X_j : $w(X_j) = 1$, $j \in \{1, \ldots, m\}$, and $w(X_0) = 2$. If $J = (j_1, \ldots, j_r)$ is a multi-index with $j_k \in \{0, 1, \ldots, m\}$, $k = 1, \ldots, r$, then define the *weight of the bracket*: $X_J = [X_{j_1}, X_K]$ where $K = (j_2, \ldots, j_r)$, by: $w(X_J) = \sum_{k=1}^{r} w(X_{j_k})$ and it's *length* by r. Then the *Hörmander's condition* is:
for every $x \in \Omega$, there exists an integer $r = r(x)$, such that

$$\dim \left(Span \left(\{ X_J(x);\ \text{length of } X_J \text{ less or equal to } r \} \right) \right) = n$$

(i.e. of maximal dimension).
For reason of precision , for any open subset $\widetilde{\Omega}$ in Ω call

$$\tau(\widetilde{\Omega}) = \sup_{x \in \widetilde{\Omega}} \inf w(X_J(x)).$$

The weight, which does not appear in the Hörmander's condition, is useful in the proof where a subelliptic estimate was proved, giving then the hypoellipticity. The index ε of subellipticity given by L. Hörmander was any real less than the inverse of $\tau(\Omega)$. There was also a simple proof by J. J. Kohn of subellipticity but with a smaller ε. Finely the exact ε of subellipticity equal to the inverse of $\tau(\Omega)$ was proved about ten years later by L. Rothshild and E. Stein .

This index is important in order to give precise results in the context of Gevrey regularity of Gevrey vectors. Let us first remark that given $x \in \Omega$ and index $\tau(x)$ given above, then there exists a small neighborhood of x, say ω such that $1/\tau(x)$ is a sub-elliptic index in ω. Recall moreover that an s-Gevrey vector of P, u, in Ω satisfies: for any compact K in Ω there exists C_K such that for any integer k one has

$$u \in L^2(K) \quad \text{and} \quad \|P^k u\|_{L^2(K)} \leq C_K^{2k+1} k^{2sk}.$$

So working with such $\tau(x)$ in ω, we got,

Theorem 2. *Assume the coefficients of P are in s-Gevrey(ω), and P satisfies the above condition. Let u an s-Gevrey vector of P in ω, then u is in $(\tau(x) \cdot s)$-Gevrey space in ω.*

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