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Arithmetic and Geometry of Low-Dimensional Algebraic Varieties
Special Session A23

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Algebraic Curves and Algebraic Surfaces are classical and fundamental topics in Algebraic Geometry, featuring profound and, at times, astonishing connections to various disciplines such as Differential Geometry, Symplectic Geometry, Complex Geometry, Commutative Algebra, and Number Theory. While many historical questions have been addressed, several open questions and challenging problems still lie before us.

The Special Session *Arithmetic and Geometry of Low Dimensional Algebraic Varieties*, scheduled on July 23-24, aims to convene numerous experts, with a specific focus on young scientists, to foster collaboration and facilitate discussions on the most recent advancements in these vital research domains.

A broad spectrum of topics will be explored, encompassing special families of curves, K3 and Abelian surfaces, surfaces of general type, Hodge structures, surfaces fibred in curves, automorphism groups, moduli spaces, rational points, and computational aspects. Special attention will be directed towards examining the interplay and connections between the arithmetic and geometric perspectives.

This Special Session will promote a robust exchange of ideas among individuals engaged in various aspects of the theory of Low-Dimensional Algebraic Varieties. Additionally, it will provide a valuable opportunity for Ph.D. students and postdocs to acquire insights into the latest results and techniques within these dynamic and rapidly evolving fields.

For more information visit
`https://sites.google.com/view/low-dimensional-varieties-2024/home`

Schedule and Abstracts

July 23, 2024

11:00–11:45 Numerically and cohomologically trivial automorphisms of elliptic surfaces

Fabrizio Catanese (Universität Bayreuth, GERMANY)

Abstract. Let X be a compact connected complex manifold: the automorphism group $\mathrm{Aut}(X)$ is a finite dimensional complex Lie Group, and we have a chain of subgroups

$$\mathrm{Aut}^0(X) \triangleleft \mathrm{Aut}_{\mathbb{Z}}(X) \triangleleft \mathrm{Aut}_{\mathbb{Q}}(X) \triangleleft \mathrm{Aut}(X), \quad \text{where}$$

i) $\mathrm{Aut}^0(X)$ is the connected component of the identity,

ii) $\mathrm{Aut}_{\mathbb{Z}}(X) := \{\sigma \in \mathrm{Aut}(X) \mid \sigma \text{ induces the trivial action on } H^*(X,\mathbb{Z})\}$ is the group of cohomologically trivial automorphisms, while the group of numerically trivial automorphisms is

iii) $\mathrm{Aut}_{\mathbb{Q}}(X) := \{\sigma \in \mathrm{Aut}(X) \mid \sigma \text{ induces the trivial action on } H^*(X,\mathbb{Q})\}$.

These subgroups are important for applications to Hodge Theory and to period mappings.

If X is Kähler, $\mathrm{Aut}_{\mathbb{Q}}(X)/\mathrm{Aut}^0(X)$ and $\mathrm{Aut}_{\mathbb{Z}}(X)/\mathrm{Aut}^0(X)$ are finite groups, and with Wenfei Liu [?] we started to investigate lower and upper bounds for the sizes of these groups in the case of surfaces S according to Kodaira dimension and in terms of the invariants of S.

There remained then the problem of studying the case of properly elliptic surfaces, i.e., minimal surfaces of Kodaira dimension one, which have a canonical elliptic fibration $f\colon S \to B$.

Theorem 1. *Assume that S is minimal of Kodaira dimension 1), with $\chi(S) = 0$: then S is isogenous to a higher elliptic product (that is, S is a free quotient quotient $(C \times E)/G$ where E is elliptic and C has genus ≥ 2).*

(I) *In the case of the pseudo-elliptic surfaces (the case where $Aut^0(S)$ is infinite, equivalently G acts by translations on E), $Aut_{\mathbb{Z}}(S) = Aut^0(S)$, except in the case*
  (I-a) *$G = \mathbb{Z}/2m$, where m is an odd integer,*
  (I-b) *$C/G = \mathbb{P}^1$ and $C \to \mathbb{P}^1$ is branched in four points with local monodromies $\{m, m, 2, -2\}$: for these we have $|Aut_{\mathbb{Z}}(X)/Aut^0(X)| = 2$.*

(II) *If $Aut_{\mathbb{Z}}(S)$ is finite and nontrivial, then it is Abelian with $|\mathrm{Aut}_{\mathbb{Z}}(S)| \leq 9$, and the case where $\mathrm{Aut}_{\mathbb{Z}}(S) \cong \mathbb{Z}/6\mathbb{Z}$ does actually occur.*

The investigation of all the possible groups occurring in (II) above requires the Reidemeister-Schreier method and heavy computations.

We discovered later that there were counterexamples to existing theorems, and also for higher values of $\chi(S)$ there was no upper bound for the order $|\mathrm{Aut}_{\mathbb{Q}}(S)|$ of the finite group $\mathrm{Aut}_{\mathbb{Q}}(S)$:

Theorem 2. *There are non-isotrivial properly elliptic surfaces S with $p_g > 0$ admitting numerically trivial automorphisms of order any given prime p.*

For $p > 5$, our examples satisfy $\chi(S) \geq (p^2 - 1)/24$.

The analysis of the case of cohomologically trivial automorphisms, and especially of upper bounds according to $\chi(S)$, is still work in progress.

References

[CatLiu21] Fabrizio Catanese and Wenfei Liu, On topologically trivial automorphisms of compact Kähler manifolds and algebraic surfaces, Rend. Lincei Mat. Appl. 32 (2021), 181–211.

[CLS24a] Fabrizio Catanese, Wenfei Liu and Matthias Schütt, On the cohomologically trivial automorphisms of elliptic surfaces–I , preprint April 2024, 25 pages.

[CLS24b] Fabrizio Catanese, Wenfei Liu and Matthias Schütt, On the numerically trivial automorphisms of elliptic surfaces–II, in preparation.

12:00–12:45 Examples of non-rigid modular vector bundles on hyperkähler manifolds.

Enrico Fatighenti (Università di Bologna, ITALY)

Abstract.

We exhibit examples of slope-stable and modular vector bundles on a hyperkähler manifold of $K3^{[2]}$-type. These are obtained by performing standard linear algebra constructions on the examples studied by O'Grady of (rigid) modular bundles on the Fano varieties of lines of a general cubic 4-fold and the Debarre-Voisin hyperkähler. Interestingly enough, these constructions are almost never infinitesimally rigid, and more precisely we show how to get (infinitely many) 20 and 40 dimensional families. This is a joint work with Claudio Onorati.

14:30–15:15 On mixed surfaces: construction and examples

Davide Frapporti (Politecnico di Milano, ITALY)

Abstract. In the last two decades there has been a growing interest in those surfaces (and varieties) birational to the quotient of a product of curves of genus at least 2 modulo the action of a finite group, and several new surfaces and varieties have been constructed in this way. These split naturally in two cases: the *unmixed* case, where each element of the group acts diagonally; and the *mixed* case, where there are elements permuting some factors besides those acting diagonally.

In the talk we will focus on surfaces and discuss the mixed case, namely: let C be a Riemann surface of genus at least 2 and $G < \mathrm{Aut}(C \times C)$ be a finite group with a mixed action (i.e., there are elements in G swapping the factors), then the quotient surface $(C \times C)/G$ is a *mixed quotient* and the minimal resolution of its singularities is a *mixed surface*. We will investigate the geometry of mixed quotients and surfaces, discuss how this is encoded in their "*algebraic data*", and present an algorithm to classify the irregular mixed surfaces with given invariants.

15:30–16:15 Correspondences acting on constant cycle curves on K3 surfaces
Sara Torelli (University of Texas at Austin, USA)

Abstract. Constant cycle curves on K3 surfaces X have been introduced by Huybrechts as curves whose points all define the Beauville-Voisin class in the Chow group of X. In this talk, we introduce correspondences $Z \subseteq X \times X$ over $\mathbb{C}$ acting on the group $\mathrm{ccc}(X)$ of cycles generated by constant cycle curves. We construct for any $n \geq 2$ and any very ample line bundle L a locus $Z_n(L) \subseteq X \times X$ of expected dimension 2, which yields a correspondence acting on the group $\mathrm{ccc}(X)$, when it has the expected dimension. We then discuss examples for low n and answer the problem of non-emptyness for $(X, |C|)$ very general pricipally polarized K3, C smooth genus g curve with general gonality and n less or equal to the general gonality. Part of the results are work in progress with Andreas Knutsen.

17:00–17:45 Higher rank lattice cohomology
Nicola Tarasca (Virginia Commonwealth University, USA)

Abstract. Lattice cohomology provides a powerful framework for constructing invariants of the link of a normal surface singularity. When the link is a rational homology sphere, these invariants can be encoded combinatorially by an infinite rooted tree. Recent results have focused on refining these invariants by introducing weights on the vertices of the graded tree. In this talk, I will show how to obtain infinitely many such invariants by considering weighted rooted trees arising from the data of a root lattice of rank at least 2. This is joint work with Allison Moore.

July 24, 2024

11:30–12:15 Tropical and logarithmic enumerative geometry of curves
Renzo Cavalieri (Colorado State University, USA)

Abstract. I will present some joint work with Hannah Markwig and Dhruv Ranganathan, in which we interpret double Hurwitz numbers as intersection numbers of the double ramification cycle with a logarithmic boundary class on the moduli space of curves. This approach removes the "need" for a branch morphism and therefore allows the generalization to related enumerative problems on moduli spaces of pluricanonical divisors - which have a natural combinatorial structure coming from their tropical interpretation. I will discuss some generalizations springing out from this approach that are currently being pursued in joint work with Hannah Markwig and Johannes Schmitt.

12:15–13:00 Irreducibility of Severi varieties on toric surfaces
Karl Christ (University of Texas at Austin, USA)

Abstract. Severi varieties parametrize integral curves of fixed geometric genus in a given linear system on a surface. In this talk, I will discuss the classical question of whether Severi varieties are irreducible and its relation to the irreducibility of other moduli spaces of curves. I will indicate how tropical methods can be used to answer such irreducibility questions. The new results are from ongoing joint work with Xiang He and Ilya Tyomkin.

14:30–15:15 Beauville varieties and groups
Christian Gleissner (Universität Bayreuth, GERMANY)

Abstract. Beauville surfaces are a certain class of rigid regular surfaces of general type that can be purely described by combinatorial methods. These surfaces play an improtant role in algebraic geometry as well as group theory, and have been studied extensively in recent years. In this presentation, I will report on a joint project with Federico Fallucca, focusing on exploring the higher-dimensional counterparts of Beauville surfaces.

15:30–16:15 Asymptotic lines on the moduli space of curves
Gian Pietro Pirola (Università di Pavia, ITALY)

Abstract. The aim is to study the second fundamental form associated with the image of the period map of curves started many years ago . We present some computational improvements that allow to study asymptotic lines in the tangent of the moduli space $\mathcal{M}_g$ of the curves of genus g. The asymptotic lines are those tangent directions that are annihilated by the second fundamental form induced by the Torelli map. We give examples of asymptotic lines for any $g \geq 4$ and we study their rank. The rank $r(v)$ of a tangent direction v at $[C] \in \mathcal{M}_g$ is defined to be the rank of the cup product map associated to the infinitesimal deformation map, that is the infinitesimal variation of Hodge structure $v : H^{1,0}(C) \to H^{0,1}(C)$, in that direction. We show that if $v \neq 0$ and $r(v) \leq \mathrm{cliff}(C)$ where $\mathrm{cliff}(C)$ is the Clifford index of C, then v is not asymptotic and we study the case when $r(v) = \mathrm{cliff}(C)$. Finally all asymptotic directions of rank 1 are determined and an almost complete description of the rank 2 case is given.
This is a report on a joint work with Elisabetta Colombo and Paola Frediani.

17:00–17:45 Surfaces with canonical map of odd degree
Rita Pardini (Università di Pisa, ITALY)

Abstract. Let S be a smooth minimal complex surface of general type. Assume that the canonical map is generically finite of degree d onto a surface Σ. By classical results of Xiao and Beauville, it is known that $d \leq 8$ if the geometric genus $p_g(S)$ is large enough, and for $d = 8$ there are examples with unbounded p_g, showing that the bound is sharp. Families of examples with $d = 2, 4, 6$ and unbounded p_g are also known, while there are only sporadic examples with $d \geq 3$ odd.

I will report on work in progress aiming at explaining this asymmetry. Under the additional assumption that the general canonical curve of S is smooth, we are able to:

(1) give an explicit lower bound on p_g in case $d = 7$;
(2) in the case when the canonical image Σ is ruled by lines, show that $p_g(S) \leq d + 2$ and Σ is a cone in $\mathbb{P}^{p_g(S)-1}$ over the rational normal curve of degree $p_g(S) - 2$.

For $d = 3$ surfaces as in (2), i.e., with Σ ruled by lines, were completely described by Mendes Lopes and Pardini and, independently, by Starnone: there are two main examples, with $p_g = 5$, $K^2 = 9$ and $K^2 = 8$, $p_g = 4$ respectively, and examples with lower invariants are obtained by degenerating these two types of surfaces.

For $d = 5$ and Σ ruled by lines a similar picture emerges from a theoretical analysis of the situation. However, quite surprisingly, we are able to rule out the existence of most of the possibilities and we have not yet been able to produce examples with $p_g > 3$.

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