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Mean Field Games and Related Topics
Special Session A1

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Mean field games are a recent and very active area of research leading to new challenging mathematical questions, from analytical, probabilistic, and numerical sides. They originated from the seminal works of Lasry and Lions, and, simultaneously, Huang, Malhamé and Caines, in 2006. In a nutshell, Mean field games represent limit models for symmetric non-cooperative N-player games, as the number of players tends to infinity. On the other hand, mean field control problems, also called optimal control of McKean-Vlasov dynamics, represent limit models for cooperative N-player games. The aim of this special session is to gather together the big community of researchers working on mean field games, mean field control and related topics mainly in Italy and in the United States.

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Schedule and Abstracts

July 23, 2024

11:00–11:45 Deterministic ergodic Mean Field Games with congestion
Martino Bardi (University of Padua, ITALY)

Abstract. I consider deterministic Mean Field Games (MFG) with a cost functional continuous with respect to the distribution of the agents and satisfying a gap condition at infinity, and compare them with the static MFG with such a cost.

Under the coercivity condition on the Hamiltonian

$$H(x,p,m) - H(x,0,m) \geq a_o|p|^\beta\,, \quad a_o > 0,\, \beta > 1\,,$$

I show how to build a solution of the ergodic MFG system of 1st order PDEs from any solution of the static MFG with cost $F(x,m) := H(x,0,m)$. This leads to new existence results under general assumptions, in particular for non-separable Hamiltonians. The motivating examples for such Hamiltonians are some models of congestion in crowd dynamics with non-local dependence on the crowd distribution, such as

$$H(x,p,m) = \frac{|p|^\beta}{(k * m(x) + \sigma)^\alpha} - F(x,m)\,, \quad \beta > 1, \sigma \geq 0, \alpha > 0.$$

Next I prove that the measure component of any solution to the ergodic MFG must solve the associated static MFG, under the following assumption on the Hamiltonian

$$H(x,p,m) - H(x,0,m) \leq a_1\, p \cdot H_p(x,p,m)\,, \quad a_1 > 0\,.$$

Such necessary condition for the solvability of the ergodic MFG implies new uniqueness results and, in some cases with multiple solutions, the characterization of all of them.

Some of these results were proved earlier for the special case $H(x,p,m) = |p|^2 - F(x,m)$ in the joint work with Hicham Kouhkouh [1], where also the asymptotics for large time of finite horizon MFG were analysed.

References

[1] M. Bardi, H. Kouhkouh, *Long-time behaviour of deterministic Mean Field Games with non-monotone interactions*, SIAM J. Math. Anal. to appear.

12:00–12:20 Convergence analysis of controlled particle systems arising in deep learning: from finite to infinite sample size

Alpar Meszaros (University of Durham, UK)

Abstract. In this talk we will consider a class of neural SDEs and discuss the limiting behavior of the associated sampled optimal control problems as the sample size grows to infinity. The neural SDEs with N samples can be linked to an N-particle system with centralized control. We analyze the Hamilton–Jacobi–Bellman equation corresponding to the N-particle system and establish regularity results which are uniform in N. The uniform regularity estimates are obtained by the stochastic maximum principle and the analysis of a backward stochastic Riccati equation. Using these uniform regularity results, we show the convergence of the minima of objective functionals and optimal parameters of the neural SDEs as the sample size $N \to +\infty$. The limiting objects can be identified with suitable functions defined on the Wasserstein space of Borel probability measures. Furthermore, quantitative algebraic convergence rates will also be discussed. The talk will be based on a joint work with H. Liao, C. Mou and C. Zhou.

12:30–12:50 Linear quadratic Mean Field Games in Hilbert spaces and some applications

Daria Ghilli (University of Pavia, ITALY)

Abstract. We study a class of linear quadratic Mean Field Games (MFG) in infinite dimension, where the state variable lives in a Hilbert space. Our motivations are problems where the state equation is a PDE or a delay equation which can be written as an ODE in a suitable Hilbert space. As a starting point, we study the case, considered in most finite dimensional contributions on the topic, where the dependence on the distribution enters just in the objective functional through the mean. This feature allows, similarly to the finite dimensional case, to reduce the usual mean field game system to a Riccati equation and a forward-backward coupled system of abstract evolution equations. Such system is completely new in infinite dimension and no results have been proved on it so far. We show existence and uniqueness of solutions for such system, applying a delicate approximation procedure.

Moreover, we study the well-posedness of the *Master equation* associated to the above mentioned MFG system and, with the help of it, we aim at proving convergence of the discrete model to the continuous one (work in progress).

We apply the results to a production output planning problem with delay in the control variable.

14:30–15:15 Stochastic Games of Intensity Control for (Ticket) Pricing

Ronnie Sircar (Princeton University, USA)

Abstract. One way to capture both the elastic and stochastic reaction of purchases to price is through a model where sellers control the intensity of a counting process, representing the number of sales thus far. The intensity describes the probabilistic likelihood of a sale, and is a decreasing function of the price a seller sets. A classical model for ticket pricing, which assumes a single seller and infinite time horizon, is by Gallego and van Ryzin (1994) and it has been widely utilized by airlines, for instance. Extending to more realistic settings where there are multiple sellers, with finite inventories, in competition over a finite time horizon is more complicated both mathematically and computationally. We discuss a dynamic mean field game of this type, and some numerical and existence-uniqueness results.

Based on work with Emre Parmaksiz.

15:30–15:50 Fourier Galerkin approximation of mean field control problems

Mattia Martini (University Côte d'Azur, FRANCE)

Abstract. The goal of this talk is to introduce a finite dimensional approximation of the solution to a mean field optimal control problem set on the d-dimensional torus, without relying on

particle-based methods. Our approximation is obtained by means of a Fourier-Galerkin method, the main principle of which is to truncate the Fourier expansion of probability measures on the torus. However, this operation has the main feature not to leave the space of probability measures invariant, which drawback is know as *Gibbs*' phenomenon.

First, we manage to prove that, for initial conditions in the 'interior' of the space of probability measures and for sufficiently large levels of truncation, the Fourier-Galerkin method actually induces a new finite dimensional control problem whose trajectories take values in the space of probability measures with a finite number of Fourier coefficients. Subsequently, our main result asserts that, whenever the cost functionals are smooth and convex, the optimal control, trajectory, and value function from the approximating problem converge to their counterparts in the original mean field control problem. Noticeably, we show that our method yields a polynomial convergence rate directly proportional to the data's regularity. This convergence rate is faster than the one achieved by the usual particles approach, offering a more efficient alternative. Furthermore, our technique also provide an explicit method for constructing an approximate optimal control along with its corresponding trajectory. This talk is based on a joint work with François Delarue.

16:00–16:20 Deep Backward and Galerkin Methods for the Finite State Master Equation

Asaf Cohen (University of Michigan, USA)

Abstract. This paper proposes and analyzes two neural network methods to solve the master equation for finite-state mean field games (MFGs). Solving MFGs provides approximate Nash equilibria for stochastic, differential games with finite but large populations of agents. The master equation is a partial differential equation (PDE) whose solution characterizes MFG equilibria for any possible initial distribution. The first method we propose relies on backward induction in a time component while the second method directly tackles the PDE without discretizing time. For both approaches, we prove two types of results: there exist neural networks that make the algorithms' loss functions arbitrarily small and conversely, if the losses are small, then the neural networks are good approximations of the master equation's solution. We conclude the paper with numerical experiments on benchmark problems from the literature up to dimension 15, and a comparison with solutions computed by a classical method for fixed initial distributions.

17:00–17:20 Regularity and quantitative stability stability for the Gibbs conditioning principle on path space via McKean-Vlasov control

Giovanni Conforti (University of Padua, ITALY)

Abstract. We consider a system of diffusion processes interacting through their empirical distribution. Assuming that the empirical average of a given observable is measured at any time, we derive regularity and quantitative stability results for the optimal solutions and control in the associated version of the Gibbs conditioning principle. The proofs rely on the analysis of a McKean-Vlasov control problem with distributional constraints. Along the way, we establish some estimates on Hamilton-Jacobi-Bellman equations and the Hessian of the log-density of diffusion processes, which are of independent interest.

17:30–17:50 Pasting of discrete time mean field equilibria and Donsker-type results for mean field games

Ludovic Tangpi (Princeton University, USA)

Abstract. In this talk we discuss mean field games in discrete time on general probability spaces. Using dynamic programming and a forward-backward algorithm, we will construct mean field equilibria of multi period models as concatenation of equilibria of one-step games. We will also present results on convergence of discrete time games to continuous time counterparts akin to Donsker's invariance principle. The talk is based on a joint work with J. Dianetti, M. Nendel and S. Wang.

July 24, 2024

11:30–12:15 Long Range Games

Paolo Dai Pra (University of Verona, ITALY)

Abstract. I will introduce a general extension of Mean Field Games that includes Kac-Potential interactions and Graphon interaction. In the context of finite state space we show that the basic properties of mean-field games continue to hold: existence of Nash equilibria, uniqueness under monotonicity, approximation of the N-player game by the limit independent games.

12:30–12:50 Self-organizing equilibria and their local stability in a Kuramoto mean field game

Annalisa Cesaroni (University of Padua, ITALY)

Abstract. Recently a Mean Field Game version of the classical Kuramoto model has been proposed in [1], describing synchronization phenomena in a large population of rational interacting oscillators. In this talk, I will discuss existence and uniqueness (up to phase transition) of the incoherent equilibrium and the self-organizing equilibrium, given that the interaction parameter is sufficiently large. Furthermore, I will also present some local stability properties of the self-organizing equilibrium with respect to dynamic equilibria in a long time regime.

References

[1] R. Carmona, Q. Cormier, H. M. Soner, *Synchronization in a Kuramoto mean field game*, Comm. Partial Differential Equations 48 (2023), no 9, 1214–1244.

14:30–15:15 Stationary Mean-field Games of Singular Control

Giorgio Ferrari (University of Bielefeld, GERMANY)

Abstract. In this talk I will present recent and ongoing results on existence, uniqueness, and characterization of equilibria for mean-field games with singular controls. This class of problems finds natural applications in Economics and Finance, such as in investment problems in oligopolies. In those games, the representative agent employs a bounded-variation control in order to maximize an ergodic profit functional depending on a long-time average of the controlled state-process. Several variants of the considered games will be presented, which will differ with respect to the dimension of the state-process and the optimality criterion employed.

15:30–15:50 Asymptotic behavior of mean field games: coercive and non coercive case

Cristian Mendico (University of Rome Tor Vergata, ITALY)

Abstract. In this talk we will revisit the recent results on the description of the asymptotic behavior of some deterministic mean field game models, namely: the classical system, the system arising from the control of acceleration and the case nonholonomic dynamics. We will discuss about the results and about the questions that are still open. Finally, we will see how the ergodic mean field game system associated with a calculus of variation problem captures the behavior of Nash equilibria.

16:00–16:20 Mean field games on homogeneous Lie groups

Claudio Marchi (University of Padua, ITALY)

Abstract. We study short-time existence of classical solutions to mean field games systems defined on homogeneous Lie groups.

More precisely, we consider an homogeneous Lie group, endowed with a family of dilations, which can be identified with $\mathbb{R}^d$. Let $\{X_1, \dots, X_m\}$ (with $m < d$) be a family of vector fields which satisfies the Hörmander condition; in particular, together with their commutators, these vector fields generate the Lie group. We consider second-order mean field games systems where the differential operators are given in terms of these vector fields and where the couplings are strongly regularizing. In our model, each agent can move only along the directions generated by $X_1, \dots, X_m$ but it can still reach every point due to the Hörmander condition.

In order to obtain existence of solution to these mean field games, we first study existence and uniqueness of the subelliptic Fokker-Planck equation and separately of the Hamilton-Jacobi equation.

17:00–17:20 Second order PDEs on the Wasserstein space

Ibrahim Ekren (University of Michigan, USA)

Abstract. We prove a comparison result for viscosity solutions of second order parabolic partial differential equations in the Wasserstein space. The comparison is valid for semisolutions that are Lipschitz continuous in the measure in a Fourier-Wasserstein metric and uniformly continuous in time. The class of equations we consider is motivated by McKean-Vlasov control problems with common noise and filtering problems. We also mention applications for prediction problems with expert advice. The proof of comparison relies on a novel version of Ishii's lemma, which is tailor-made for the class of equations we consider.

17:30–18:15 Synchronization Games

Mete Soner (Princeton University, USA)

Abstract. Building on Winfree's work, the Kuramoto model (1975) has become the corner stone of mathematical models of collective synchronization, and has received attention in all natural sciences, engineering, and mathematics. While the classical model postulates the dynamics of each oscillator in the form of a system of nonlinear ordinary differential equations, Yin, Mehta, Meyn, and Shanbhag (2010) use the mean-field game (MFG) formalism of Lasry and Lions, and Huang, Caines, and Malhamé. In this talk, in addition to the Yin et al model, we also introduce a simpler two state model which can be seen as a discretization of the original one. We outline results showing that the mean field approach also delivers same type of results including the phase transition from incoherence to synchronization. In particular, in the discrete setting we provide a comprehensive characterization of stationary and dynamic equilibria along with their stability properties. In all models, while the system is unsynchronized when the coupling is not sufficiently strong, fascinatingly, they exhibit an abrupt transition to a full synchronization above a critical value of the interaction parameter. In the subcritical regime, the uniform distribution representing incoherence is the only stationary equilibrium. Above the critical interaction threshold, the uniform equilibrium becomes unstable and there is a multiplicity of stationary equilibria that are self-organizing. The discrete model with discounted cost present dynamic equilibria that spiral around the uniform distribution before converging to the self-organizing equilibria. With an ergodic cost, however, unexpected periodic equilibria around the uniform distribution emerge.

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