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Free boundary problems: viscosity and variational approaches Special Session B9

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This session is scheduled on July 25-26 and it is focused on Free boundary problems (FBP).

This session is scheduled on July 25-26 and it is focused on **Free boundary problems (FBP).** Free boundary problems are a type of differential equation where the domain in which the equation holds, depends itself on the solution. Examples arise in flame propagation, image reconstructions, jet flows, optimal stopping problems in financial mathematics, tumor growth, and in many other different contexts, providing the opportunity for an interplay between the applied sciences and pure mathematical analysis.

A classical example, which we will use here to illustrate the significance of FBP, is the Bernoulli problem. It arises in two dimensional fluid dynamics and it was first studied systematically by Alt and Caffarelli using a variational approach, in the one-phase setting i.e for non-negative solutions. The two-phase case, in which a solution is allowed to change sign, was then investigated by Alt, Caffarelli, and Friedman who devised a fundamental monotonicity formula. The theory was propelled forward by breakthrough results due to Caffarelli and relying on a viscosity formulation of the problem which allowed to take a geometric approach to answer the essential question of the regularity of the free boundary.

Other analogous problems arise when considering models for which the free boundary can occur on a lower dimensional space, for example the *thin Bernoulli problem*. Related problems also appear in the study of cooperative systems of species, in optimization problems for spectral functions, in optimal partition problems, or in the study of harmonic functions with junctions Several evolution problems are also connected to the Bernoulli problem, like the Stefan problem or the Hele-Shaw problem used to describe an incompressible flow lying between two nearby horizontal plates Furthermore, one can consider FBP in non-commutative structures. Other fundamental problems are obstacle-type problems, which also present a vast literature.

Given the interest of the international scientific community for free boundary problems, our special session is dedicated to this topic. In particular, the invited speakers take different approaches to attack important and current questions, whether with variational or viscosity tools.

In the effort of fostering collaborations, especially between senior and junior researchers, we will advertise the session to mathematicians at all levels, at different institutions both in the U.S and in Italy.

This two-day session will feature several lectures of varying length. The special session is upported by the Department of Mathematics of the university of Bologna and INDAM

The list of the speakers who accepted to deliver a seminar and further information are available at the following working in progress web page:
https://math.unibo.it/en/events/free-boundary-problems-viscosity-and-variational-approaches

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Schedule and Abstracts

July 25, 2024, Morning Session

11:30–12:15 Sharp order of vanishing for parabolic equations, nodal set estimates and Landis type results

Nicola Garofalo (University of Padua, ITALY)

Abstract. I will present a best possible estimate of the order of vanishing of solutions to parabolic equations. For real-analytic leading coefficients, we establish a localised estimate of the nodal set, at a given time-level, that generalises the celebrated one of Donnelly and Fefferman. We also obtain Landis type results for global solutions. This is joint work with Vedansh Arya & Agnid Banerjee.

12:30–12:50 Boundary behavior of solutions to fractional elliptic problems

Serena Dipierro (University of Western Australia, AUSTRALIA)

Abstract. Solutions of nonlocal equations typically depend rather significantly on their values outside of a given region of interest and, in this sense, it is often convenient to assume "global" conditions to deduce "local" results. In this talk, we present instead a Hopf Lemma for solutions to some integro-differential equations that does not assume any global condition on the sign of the solutions. We also show that non-trivial radial solutions cannot have infinitely many zeros accumulating at the boundary.

July 25, 2024, Afternoon Session-Part I

2:30–3:15 p.m. A two-phase free boundary problem for an operator with non-standard growth

Claudia Lederman (University of Buenos Aires and CONICET, ARGENTINA)

Abstract. We will present recent results on a two-phase free boundary problem for an operator with non-standard growth.

In fact, we consider viscosity solutions to a free boundary problem for a nonlinear elliptic PDE with non-zero right hand side. We obtain regularity results for solutions and their free boundaries.

The study of PDE's of the type considered is motivated by their application in the modelling of different phenomena, such as non-Newtonian fluids, non-linear elasticity and image reconstruction.

The nonlinear degenerate/singular nature of these equations leads to challenging difficulties that will be discussed in this talk.

The fact that our results hold for merely viscosity solutions allows a wide applicability. This is joint work with Fausto Ferrari (University of Bologna).

3:30–3:50 p.m. Boundary unique continuation problems and size estimates of critical sets

Stefano Vita (University of Turin, ITALY)

Abstract. In this talk we discuss some recent results on boundary unique continuation problems. In particular, we describe new techniques to obtain size estimates of singular and critical sets of harmonic functions up to a boundary where the functions vanish.

4:00–4:20 p.m. Free boundary regularity in an optimal partition problem

Roberto Ognibene (University of Pisa, ITALY)

Abstract. Let us consider a bounded domain, divided into a fixed number of disjoint subdomains and, among all the possible configurations, let us consider the one for which the sum of the first Dirichlet eigenvalues of the subdomains is minimal. In this talk, I will discuss the regularity of the interface which emerges as boundary of such optimal partition and, in particular, I will focus on the regularity up to the fixed boundary. The talk is based on joint works with B. Velichkov.

July 25, 2024, Afternoon Session-Part II

5:00–5:45 p.m. Fully nonlinear equations in thin domains: a test function approach

Isabeau Birindelli (Sapienza University of Rome, ITALY)

Abstract. In this talk I will present a work done in collaboration with Ariela Briani and Hitoshi Ishii, where we extend to fully nonlinear operators the well known result on thin domains of Hale and Raugel. In 1992 they proved that solutions of a Neumann problem in thin domains

$\Omega_\epsilon = \{(x, y) \in R^N \times R \mid x \in \Omega,\ 0 < y < \epsilon g(x)\}$, will converge for ϵ going to zero to the solution of a Neumann problem in Ω where the equation itself has an extra first order term coming from the top boundary $y = g(x)$. With a totally new approach, instead of the Laplacian we consider fully nonlinear operators that are proper in the sense of the User's guide but may be degenerate elliptic. The result is more general even in the case of the Laplacian.

6:00–6:20 p.m. On some segregation models

Stefania Patrizi (University of Texas at Austin, USA)

Abstract. Segregation phenomena occur in many areas of mathematics and science: from equipartition problems in geometry, to social and biological processes (cells, bacteria, ants, mammals) to finance (sellers and buyers). Segregation problems model a situation of high competition for resources and involve a combination of diffusion and annihilation between populations. We present three different models: in the first one the competition between species is nonlocal, meaning that the growth of a population at a point is inhibited by all other populations in a full neighbor of that point; in the second one the diffusion terms are given by fully nonlinear operators; in the third one two competing species follow propagation equations, one of them involving a local diffusion while the other one involving a non-local diffusion. These are joint works with Luis Caffarelli, Veronica Quitalo and Monica Torres.

July 26, 2024, Morning Session

11:30–12:15 Boundary Harnack principles and Schauder estimates for degenerate equations on singular sets

Susanna Terracini* (University of Turin, ITALY)

Abstract The ratiov/u of two solutions to a second order elliptic equation in divergence form solves a degenerate elliptic equation if u and v share the zero set; that is, $Z(u) \subseteq Z(v)$. The coefficients of the degenerate equation vanish on the nodal set as u^2. Developing a Schauder theory for such equations, we prove $C^{k,\alpha}$-regularity of the ratio from one side of the regular part of the nodal set in the spirit of the higher order boundary Harnack principle established by De Silva and Savin in [4]. Then, by a gluing lemma, the estimates extend across the regular part of the nodal set. Eventually, using conformal mapping in dimension $n = 2$, we provide local gradient estimates for the ratio which hold also across the singular part of the nodal set and depends on the highest value attained by the Almgren frequency function.

*Joint work with Giorgio Tortone and Stefano Vita.

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12:30–12:50 Plateau's laws for surfaces of minimal capacity

Daniel Restrepo (Johns Hopkins University, Baltimore, USA)

Abstract. In this talk, we will discuss a novel family of free boundary problems that arise in the study of variational problems with topological constraint in their level sets. More precisely, given a compact set $\mathcal{W} \subset \mathbb{R}^{n+1}$ (a "wire frame"), and a potential $F : [0, 1] \to [0, \infty)$, we consider the minimization problems

$$\inf \Big\{ \int_\Omega |\nabla u|^2 + F(u) \,:\, u \in C_0^1(\Omega; [0, 1]) \,,\, \{u = 1\} \text{ spans } \mathcal{W} \Big\} \,, \tag{1}$$

where $\Omega = \mathbb{R}^{n+1} \setminus \mathcal{W}$. In the case where $F(t)$ is a double-well potential, solutions of (1) corresponds to Allen-Cahn approximations to soap films spanning the wire frame $\mathcal{W}$, whereas if $F \equiv 0$, (1) models surfaces of minimum capacity attached to $\mathcal{W}$.

The heart of the matter in this reformulation of various classical variational problems is the notion of spanning, which corresponds to the so-called homotopic spanning condition introduced in [2] in the context of Plateau's problem -and extended to Sobolev functions in [3]. Homotopic spanning has the advantage of being compatible to the so-called Plateau's laws, meaning that minimizers to Plateau's problem with such boundary conditions are analytic except at some special conical singularities (see [1,4]). We will see that a similar property holds for minimizers to (1): the free boundary $\{u = 1\}$ is a smooth surface possibly except at some special conical singularities including, for instance, triple junctions. Aside of this novel aspect of the singular set, we will discuss in what extent the free boundary problem associated with (1) still shares several interesting features with other well-known models, like the two-phase Bernoulli and optimal partition problems.

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July 26, 2024, Afternoon Session-Part I

2:30–3:15 p.m. Regularity results for a penalized thin obstacle problem with variable coefficients

Donatella Danielli (Arizona State University, USA)

Abstract. In this talk we will discuss a two-penalty boundary obstacle problem for a divergence form elliptic operator, motivated by applications to fluid dynamics and thermics. Our goals are to establish regularity properties of the solution and structural properties of the free boundary. The proofs are based on tailor-made monotonicity formulas of Almgren, Weiss, and Monneau type, combined with the classical theory of oblique derivative problems. This is joint work with Brian Krummel (University of Melbourne).

3:30–3:50 p.m. A capillarity one-phase Bernoulli free boundary problem

Giorgio Tortone (University of Pisa, ITALY)

Abstract. This seminar explores one-phase Bernoulli free boundary problems within smooth containers, focusing on scenarios where the normal derivative of solutions is prescribed along the containers boundary. We examine the regularity of the free boundary and the structure of the wetting region, characterized by its contact with the fixed boundary.
Key findings include characterizing the contact angle in terms of container permeability and unveiling the smooth $(d-2)$-dimensional nature of the boundary of the wetting region, except for a possible closed set of dimension at most $d-5$.

This is based on a joint work with L. Ferreri and B. Velichkov.

4:00–4:20 p.m. Regularity for almost minimizers of a degenerate Bernoulli-type functional

Nicolò Forcillo (Michigan State University, USA)

Abstract. In this talk, we deal with almost minimizers of the energy functional

$$J_p(u,\Omega) := \int_\Omega \Big(|\nabla u(x)|^p + \chi_{\{u>0\}}(x)\Big)\, dx, \quad p>1, \tag{2}$$

where Ω is a bounded domain in $\mathbb{R}^n$ and $u \geq 0$. The functional J_p is a generalization to each $p>1$ of the classical one-phase (Bernoulli) energy functional ($p=2$ in (2)).
Almost minimizers of J_2 were investigated in [2,1]. However, D. De Silva and O. Savin provided in [4] a different approach than [2,1], based on nonvariational techniques, to study almost minimizers of J_2 and their free boundaries. Precisely, inspired by [5], they showed that almost minimizers of J_2 are "viscosity solutions" in a more general sense. This property roughly means that almost minimizers satisfy comparison in a neighborhood of a touching point whose size depends on the properties of the test functions. Once this fact was established, the regularity of the free boundary for almost minimizers followed via the techniques developed by De Silva in [3].
In this talk, we present an optimal Lipschitz continuity result for almost minimizers of J_p, with $p > \max\left\{\frac{2n}{n+2}, 1\right\}$. Our approach is inspired by [4]. Our method mostly relies on using p-harmonic replacements as competitors. The regularity properties of these replacements indeed allow us to infer the Lipschitz continuity of almost minimizers. In particular, we first prove a dichotomy-type result, and next, we improve and iterate one of the two alternatives of the dichotomy. The talk is based on joint work with S. Dipierro, F. Ferrari, and E. Valdinoci, see [6].

References

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July 26, 2024, Afternoon Session-Part II

5:00–5:45 p.m. Flipping one sided regularity via a Harnack approach and applications to nonlinear elliptic problems

Diego Moreira (Universidade Federal do Ceará, BRAZIL)

Abstract. In this talk we discuss some recent advances on the regularity theory of non linear elliptic problems showing that weak Harnack type arguments allow the passage from one-side regularity to full regularity in Hölder and Sobolev spaces. As a particular case of these phenomena, we can identify the Caffarelli, Kohn, Nirenberg and Spruck theorem (in the 80s), as well as, some more recent regularity results obtained together with Alessio Figalli (ETH) and Ederson Braga (UFC) both on the regularity of semiconvex supersolutions of uniformly elliptic equations. This problem has some motivations linked to free boundary problems as well. This is a joint work with Edgard Pimentel (University of Coimbra).

6:00–6:20 p.m. Regularity for almost minimizer with free boundary in Carnot groups of step two

Enzo Maria Merlino (University of Bologna, ITALY)

Abstract. The regularity of minimizers of the classical one-phase Bernoulli functional was deeply studied after the pioneering work of Alt and Caffarelli. More recently, the regularity of almost

minimizers was investigated as well. We present a regularity result for almost minimizers for a one-phase Bernoulli-type functional in Carnot Groups of step two. Our approach is inspired by the methods introduced by De Silva and Savin in the Euclidean setting. Moreover, some recent intrinsic gradient estimates have been employed. Generalizations to the nonlinear framework will be discussed. Some of the results presented are obtained in collaboration with F. Ferrari (University of Bologna) and N. Forcillo (Michigan State University).

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