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Discrete and Combinatorial Algebraic Topology, Theory and Applications
Special Session B16

Bruno Benedetti
University of Miami, USA

Anton Dochtermann
Texas State University, USA

Chris Kapulkin
University of Western Ontario, CANADA

Giovanni Paolini
Università di Bologna, ITALY

Antonio Rieser
Centro de Investigación en Matemáticas, A.C., MEXICO

Extending techniques from algebraic topology to discrete and combinatorial objects such as graphs and finite posets has recently generated an increased amount of interest from researchers working in a variety of areas of mathematics. Several different parts of algebraic topology have undergone at least a partial 'discretization' so far, the most notable being homotopy theory, Morse theory, and sheaf theory, and efforts to make the discrete theories more complete are ongoing. The work has already found numerous applications, including to hyperplane arrangements, dynamical systems, geometric group theory, coarse geometry, configuration spaces, distributed computing, graph colorings, and digital imaging, as well as to network and data analysis. The goal of this session is to bring together researchers who are currently working in discrete algebraic topology with researchers who work in closely related areas in order to expand the mathematical applications of discrete-algebraic-topological techniques and disseminate recent developments in the field.

For more information visit `https://sites.google.com/cimat.mx/umi-ams-discrete-alg-top`

Schedule and Abstracts

July 25, 2024

11:30–11:50 Combinatorics and Topology
Caroline Klivans (Brown University, USA)
Abstract. TBD

12:00–12:20 TBD
Eric Babson (University of California, Davis, USA)
Abstract. TBD

12:30–12:50 On the Common Basis Complex and the Partial Decompositions Poset
Volkmar Welker (Universität Marburg, Germany)
Abstract. In this talk we discuss joint work with B. Brück and K. Piterman on generalizations of the common basis complex studied in algebraic K-theory. There for a ring R the common basis complex $\mathrm{CB}(R^n)$ is the simplicial complex whose simplices are collections of free summands $\neq 0, R^n$ for which there is a common basis. The dimension of $\mathrm{CB}(R^n)$ is $2^n - 3$ but there is a conjecture due to Rognes that for a large class of rings the complex if $2n - 4$ connected and it is know that again for a large class of rings the homology above dimension $2n - 3$ vanishes. For fields Rognes conjecture was recently proved by Miller, Patzt and Wilson.

We show in the generality of our setting that the common basis complex is indeed homotopy equivalent to the order complex of partial decompositions. The latter poset for the case of R^n consists of all sets of free submodules that are in direct sum and for which their direct sum has a free complement. It is $2n-3$ dimensional and hence Rognes conjecture suggests that it could be Cohen-Macaulay. In our generality this is not the case, but it seems to hold for many examples.

References

[1] B. Brück, K. Piterman, V. Welker, The common basis complex and the poset of partial decompositions, https://arxiv.org/pdf/2402.10484

2:30–2:50 Topological methods in zero-sum Ramsey theory

Florian Frick (Carnegie Mellon University, USA)

Abstract. A cornerstone result of Erdős, Ginzburg, and Ziv states that any sequence of $2n-1$ elements in $\mathbb{Z}/n$ contains a zero-sum subsequence of length n. This result has inspired numerous generalizations and variants, collectively known as zero-sum Ramsey theory. In a general form, these results give conditions for any $\mathbb{Z}/n$-labelling of the vertices of an n-uniform hypergraph to have a hyperedge whose labels sum to zero. I will present a topological condition for this to occur in terms of the box complex of the hypergraph. This topological condition for the special case of Kneser hypergraphs is implied by a purely combinatorial criterion in terms of the colorability defect. This provides a unified topological approach to earlier results, such as the original theorem of Erdős, Ginzburg, and Ziv, Olson's generalization to arbitrary finite groups, and zero-sum matchings in hypergraphs. It also yields new zero-sum Ramsey results.

References

[1] F. Frick, J. Lehmann Duke, M. McNamara, H. Park-Kaufmann, S. Raanes, S. Simon, D. Thornburgh, and Z. Wellner, *Topological methods in zero-sum Ramsey theory*, arxiv:2310.17065 (2023).

3:00–3:20 Homotopy and singular homology groups of finite (di)graphs

Nikola Milićević (Pennsylvania State University, USA)

Abstract. We extend classical results in algebraic topology for higher homotopy groups and singular homology groups of pseudotopological spaces. Pseudotopological spaces are a generalization of topological spaces that also include simple directed and undirected graphs. More specifically, we show the existence of a long exact sequence for homotopy groups of pairs of closure spaces and that a weak homotopy equivalence induces isomorphisms for homology and cohomology groups.

Theorem 1. Given a pointed pair of pseudotopological spaces (X, A, x_0), the sequence

$$\cdots \to \pi_n(A, x_0) \xrightarrow{i_*} \pi_n(X, x_0) \xrightarrow{j_*} \pi_n(X, A, x_0) \xrightarrow{\partial} \cdots \to \pi_1(X, A, x_0) \xrightarrow{\partial} \pi_0(A, x_0) \xrightarrow{i_*} \pi_0(X, x_0),$$

is exact, where i_*, j_* are the maps induced my inclusions $i : (A, x_0) \to (X, x_0)$ and $j : (X, x_0, x_0) \to (X, A, x_0)$, respectively and ∂ is the boundary operator described above.

Theorem 2. A weak homotopy equivalence $f : X \to Y$ of pseudotopological spaces induces isomorphisms $f_* : H_n(X; G) \to H_n(Y; G)$ and $f^* : H^n(Y; G) \to H^n(X; G)$ for all n and all coefficient groups G.

With these results we are able to prove out main result, the construction of a weak homotopy equivalences between the geometric realizations of (directed) clique complexes and their underlying (directed) graphs.

Theorem 3. For each finite directed (resp. undirected) graph (X, E) there exist a finite abstract simplicial complex $\overrightarrow{\mathrm{VR}}(X, E)$ (resp. $\mathrm{VR}(X, E)$) and a weak homotopy equivalence $f_X : |\overrightarrow{\mathrm{VR}}(X, E)| \to (X, E)$ (resp. $f_X : |\mathrm{VR}(X, E)| \to (X, E)$) in **PsTop**.

This implies that singular homology groups of finite graphs can be efficiently calculated from finite combinatorial structures, despite their associated chain groups being infinite dimensional. This work is similar to the work McCord [1] did for finite topological spaces, but in the context of pseudotopological spaces. Our results also give a novel approach for studying (higher) homotopy groups of discrete mathematical structures such as digital images.

References

[1] M.C. McCord, *Singular homology groups and homotopy groups of finite topological spaces*, (1966), 465-474.

3:30–3:50 Hardness of Promised Colourings and Homotopy of Relational Structures

Marek Filakovský (Masaryk University, Czech Republic)

Abstract. The *constrain satisfaction problem* (CSP) is a problem of finding a homomorphism of relational structures and can be stated as follows: Let A be a fixed relational structure (i.e. a set together with a list of relations of various arities), then CSP(A) asks whether a given relational structure X admits a homomorphism $X \to A$. We remark that the classical *graph colouring problem* can be seen as a special case of CSP and that deciding whether a graph is k-colourable (CSP(K_k))is NP-complete for $k \geq 3$ [2].

The framework of CSP can further be extended to a *promised* version (PCSP). Here, we fix two strucutes A, B with a known homomorphism $A \to B$. PCSP(A, B) then asks whether for a given structure X one can either find a homomorphism $X \to A$ or at least show there is no homomorphism $X \to B$. Again, there is a related special case in the theory of graphs:

Given an input graph that is *promised* to be c-colourable, how hard is it to find at least a k-colouring, $k \geq c \geq 3$? This problem (formally PCSP(K_c, K_k)) is conjectured to be NP-hard, with only several cases proven so far [3].

A common technique in the study of a (P)CSP instance is to consider the structure of *polymorphisms* i.e. homomorphisms $A^n \to B$. In recent years, a topological approach for studying the polymorphism was introduced by Krokhin, Opršal, Wrochna, and Živný [4].

In the talk, I'll give an overview of the topological approach, stress the role of various versions of homotopy for relational structures and finally demonstrate an application of the homotopy theory methods in in the study of *linearly ordered* colourings of hypergraphs (joint w. J. Opršal, T. Nakajima, G. Tasinato and U. Wagner [1]):

A linearly ordered (LO) k-colouring of a hypergraph is a colouring of its vertices with colours $1, \dots, k$ such that each edge contains a unique maximal colour.

We prove that the following promise problem is NP-complete: Given a 3-uniform hypergraph, distinguish between the case that it is LO 3-colourable, and the case that it is not even LO 4-colourable.

References

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4:00–4:20 The connectivity of Vietoris-Rips complexes of spheres

Henry Adams (University of Florida, USA)

Abstract. For X a metric space and $r > 0$, the Vietoris-Rips simplicial complex $VR(X; r)$ contains X as its vertex set, and a finite subset of X as a simplex if its diameter is less than r. Some versions of discrete homotopy groups are closely related to the standard homotopy groups of Vietoris–Rips complexes. Interestingly, the Vietoris–Rips complexes of the circle obtain the homotopy types of the circle, the 3-sphere, the 5-sphere, the 7-sphere, ..., as the scale parameter increases. But little is known about Vietoris-Rips complexes of the n-sphere S^n for $n \geq 2$. We

show how to control the homotopy connectivity of Vietoris-Rips complexes of spheres in terms of coverings of spheres and projective spaces. For $\delta > 0$, suppose that the first nontrivial homotopy group of $VR(S^n; \pi - \delta)$ occurs in dimension k, i.e., suppose that the connectivity is $k - 1$. Then there exist $2k + 2$ balls of radius δ that cover S^n, and no set of k balls of radius $\delta/2$ cover the projective space $\mathbb{R}P^n$.

5:00–5:20 Überhomology, dominating sets and the Mayer-Vietoris spectral sequence

Luigi Caputi (University of Bologna, Italy)

Abstract. Überhomology is a recently defined triply-graded homology theory of simplicial complexes, which yields both topological and combinatorial information. When restricted to (simple) graphs, a certain specialization of überhomology gives a categorification of the connected domination polynomial at -1; which shows that überhomology of graphs is related to combinatorial quantities such as connected dominating sets. On the topological side, überhomology detects the fundamental class of homology manifolds. In this talk, we present the notion of überhomology, as introduced by D. Celoria, and show some combinatorial and topological properties such as its relation to connected domination. Then, we shall see a more conceptual property: überhomology of simplicial complexes can be identified with the second page of the Mayer-Vietoris spectral sequence, with respect to the anti-star covers.

5:30–5:50 Telling apart coarsifications of the integers

Federico Vigolo (Georg-August Universität Göttingen, Germany)

Abstract. A *coarse homomorphism* between two groups equipped with bi-invariant metrics is a Lipschitz mapping that commutes with the group operation up to bounded error. The group $\mathbb{Z}$ of integers can be given a moltitude of (non proper) invariant metrics that are not bi-Lipschitz equivalent to one another. Now the question is: if we take two copies of $\mathbb{Z}$ equipped with two such metrics, is there a coarse isomorphism between them? In this talk I will discuss an invariant of coarse isomorphism and show how to use it to answer this question for word metrics associated with certain geometric progressions. This is part of more general investigations on the topic of *coarse groups.*

References

[1] L. Schäfer and F. Vigolo, *On a coarse invertibility spectrum for coarse groups*, Preprint (2024) arxiv.org/abs/2404.08536.

[2] A. Leitner and F. Vigolo, *An Invitation to Coarse Groups*, Springer Lecture Notes in Mathematics, **2339**, (2023).

6:00–6:20 On Combinatorial Width and the Homeomorphism Problem for 3-Manifolds

Kristóf Huszár (Graz University of Technology, Austria)

Abstract. The d-dimensional *Homeomorphism Problem* HP_d asks whether two given closed, orientable, triangulated d-manifolds are homeomorphic. Across the dimensions, the difficulty of HP_d is strikingly different. While HP_2 is easily solved, for any fixed $d \geq 4$ there is no general algorithmic solution to HP_d. In the remaining dimension $d = 3$, the Homeomorphism Problem turns out to be algorithmically decidable, but the known solutions are very complicated and have not been implemented.

Thus, in practice, HP_3 is usually approached with the help of various topological invariants computable from triangulations. (Un)fortunately, those invariants that are fine enough to distinguish a large number of 3-manifold triangulations are often known to be very hard to compute in general. In the last decade, however, it has been shown that some of these invariants can actually be computed very efficiently for triangulations that are "thin" in a combinatorial sense.

In this talk, we present several recent results that relate the key combinatorial width parameter in the above context (the treewidth of the dual graph of a 3-manifold triangulation) to classical topological invariants of 3-manifolds (such as the Heegaard genus or the JSJ decomposition). As a consequence of these results, we exhibit infinite families of 3-manifolds that do not admit "thin" triangulations.

Joint work with Jonathan Spreer and Uli Wagner.

References

[1] Kristóf Huszár and Jonathan Spreer, *On the width of complicated JSJ decompositions*, 39th International Symposium on Computational Geometry, Art. No. 42, 18, Leibniz International Proceedings in Informatics (LIPIcs), volume 238, 2023, doi:10.4230/lipics.socg.2023.42.

[2] Kristóf Huszár and Jonathan Spreer, *3-manifold triangulations with small treewidth*, 35th International Symposium on Computational Geometry, Art. No. 44, 20, Leibniz International Proceedings in Informatics (LIPIcs), volume 129, 2019.

[3] Kristóf Huszár, Jonathan Spreer, and Uli Wagner, *On the treewidth of triangulated 3-manifolds*, J. Comput. Geom. 10(2):70-98, 2019, doi:10.20382/jocg.v10i2a5.

July 26, 2024

11:30–11:50 Symmetry counts: an introduction to equivariant Hilbert and Ehrhart series

Alessio D'Alì (Politecnico di Milano, Italy)

Abstract. The Ehrhart series of a lattice polytope P is a combinatorial gadget that counts the number of lattice points of P and of its dilations. The Hilbert series of a simplicial complex Σ counts how many monomials supported on faces of Σ exist in each possible degree. The aim of this talk is to introduce equivariant versions of such constructions, where we are not just interested in counting but we also want to record how the action of a finite group affects such collections of lattice points or monomials. Inspired by previous results by Betke–McMullen, Stembridge, Stapledon and Adams–Reiner, we will investigate which extra combinatorial features of the group action give rise to "nice" rational expressions of the equivariant Hilbert and Ehrhart series, and how the two are sometimes related.

References

[1] A. Adams and V. Reiner, *A colorful Hochster formula and universal parameters for face rings*, J. Commut. Algebra, 15 (2023), 151–176.

[2] U. Betke and P. McMullen, *Lattice points in lattice polytopes*, Monatsh. Math., 99 (1985), 253–265.

[3] A. Stapledon, *Equivariant Ehrhart theory*, Adv. Math., 226 (2011), 3622–3654.

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[5] J. R. Stembridge, *Some permutation representations of Weyl groups associated with the cohomology of toric varieties*, Adv. Math., 106 (1994), 244–301.

12:00–12:20 Eulerian magnitude homology

Giuliamaria Menara (Università degli Studi di Trieste, Italy)

Abstract. Magnitude was first introduced by Leinster in 2008 as a notion analogous to the Euler characteristic of a category. Magnitude homology was defined in 2014 by Hepworth and Willerton as a categorification of magnitude in the context of simple undirected graphs, and although the construction of the boundary map suggests that magnitude homology groups are strongly influenced by the graph substructures, it is not straightforward to detect such subgraphs.

In this talk, I will describe the work done by Chad Giusti and myself toward elucidating the connection between magnitude homology of simple graphs equipped with the hop metric and their combinatorial structure. The approach we take is to observe that a large portion of the magnitude chain complex is redundant, in the sense that the chains reflect combinatorial structure already recorded by chains of lower bigrading. To leverage this observation, we define the subcomplex of *eulerian magnitude chains* $EMC_{k,\ell}(G)$, supported on trails with no repeated landmarks. Focusing on the $k = \ell$ line where the list of landmarks completely determines a trail, we obtain strong relationships between the (k, k)-eulerian magnitude homology groups and the structure of a graph. We accomplish this in Theorem 1 by decomposing such cycles into a generating set described in terms of their *structure graphs*, which encode how terms in the differentials of the constituent chains cancel. We are thus able to characterize subgraphs of a

graph that support non-trivial cycles in $EMH_{k,k}(G)$ in terms of the corresponding structure graphs, providing a framework for computing these groups for graphs of interest.

Theorem 1. *Let G be a graph and $X = \{\bar{x}^i\}_{i\in[m]} \subseteq EMC_{k,k}(G)$ a collection of trails in G. Then we can decompose EMH cycles supported on X into cycles supported on cliques-trees and cycles supported on circuits of the structure graph $s(X)$.*

In the interest of exploring what features of a graph the (eulerian) magnitude homology groups capture, we turn our attention to classes of random graphs: Erdős-Rényi (ER) random graphs and random geometric graphs on the standard torus. In each context, we derive a vanishing threshold for the limiting expected rank of the (k, k)-eulerian magnitude homology in terms of the density parameter. Further, adapting tools from Kahle and Meckes we develop a characterization of the limiting expected Betti numbers of the (k, k)-eulerian magnitude homology groups in terms of density. In this talk, I will focus on the results in the context of ER graphs.

Theorem 2. *Let $G(n, p)$ be an ER graph and call $\beta_{k,k} = \mathrm{rank}(EMH_{k,k}(G(n, p)))$. If $p = o\left(n^{-1/n}\right)$ then $\lim_{n\to\infty} \frac{\mathbb{E}(\beta_{k,k})}{n^{k+1}p^{2k-1}} = \frac{1}{(k+1)!}$ and $\frac{\beta_{k,k}-\mathbb{E}(\beta_{k,k})}{\sqrt{Var(\beta_{k,k})}} \Rightarrow N(0, 1)$.*

I will finally discuss the homotopy type of the eulerian magnitude chain complex by highlighting its connection with the complex of injective words.

References

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12:30–12:50 Bigraded path homology and the magnitude-path spectral sequence

Emily Roff (University of Edinburgh, Scotland)

Abstract. Magnitude homology, path homology and reachability homology are all homology theories for directed graphs, each satisfying sensible analogues of the Eilenberg–Steenrod axioms. In spite of this, they tend to disagree even on very primitive classes of graphs. For example, consider the directed cycle Z_m of m vertices and m edges, consistently oriented. Whereas magnitude homology distinguishes Z_m from Z_n whenever $m \neq n$, path homology sees Z_1 and Z_2 as "contractible" and all the rest as "circle-like". Meanwhile, from the perspective of reachability homology, every directed cycle appears contractible.

The *magnitude-path spectral sequence* (or *MPSS*) offers a systematic account of these different points of view. Page E^1 of the MPSS is exactly magnitude homology, while (as Asao showed in [1]) path homology can be identified with a single axis of page E^2; the sequence converges, under mild conditions, to reachability homology [3]. In [4] we name page E^2 as a whole the *bigraded path homology* of a directed graph, and study its properties along with those of the other pages of the MPSS. For each $r \in \mathbb{N}$, page E^{r+1} of the MPSS has a homotopy invariance property that holds when maps of directed graphs $f, g\colon G \to H$ are *r-close* with respect to the shortest path metric. (In particular, page E^r sees the directed cycle Z_m as contractible when m is less than r, and distinguishes each of the Z_ms for $m \geq r$.) We demonstrate that each page of the sequence possesses homological properties compatible with this ever-stronger homotopy invariance. Thus, the MPSS encompasses a spectrum of homological perspectives on the category of directed graphs, interpolating between magnitude homology and reachability homology.

Concerning the spectral sequence as a whole, our main results are as follows.

Theorem 1. *Every page of the MPSS has the following homological properties:*

- *It satisfies an excision theorem with respect to the cofibrations introduced in* [2].

- *It satisfies a Künneth theorem with respect to the box product.*
- *It is a finitary functor on the category of directed graphs.*

In particular these hold for bigraded path homology, which also satisfies a Mayer-Vietoris theorem.

Together, these results allow us to show that the cofibration category structure exhibited by Carranza *et al* in [2] admits a natural refinement.

Theorem 2. *The category of digraphs carries a cofibration category structure in which the cofibrations are those of* [2] *and the weak equivalences are maps inducing isomorphisms on bigraded path homology. This is a strictly finer structure than the one exhibited in* [2]*: for instance, bigraded path homology, unlike path homology, distinguishes the directed m-cycles for every $m \geq 2$.*

We speculate that this cofibration category belongs to a nested family of structures, one for each page of the MPSS. Time permitting, I will sketch this idea at the end of the talk.

References

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2:30–2:50 Threshold-linear networks, attractors, and oriented matroids
Carina Curto (Brown University, USA)

Abstract. Threshold-linear networks (TLNs) are common models in theoretical neuroscience that are useful for modeling neural activity and computation in the brain. They are simple, recurrently-connected networks with a rich repertoire of nonlinear dynamics including multistability, limit cycles, quasiperiodic attractors, and chaos. In this talk, I will give a brief introduction to TLNs, and then show how ideas from sheaf theory and oriented matroids provide valuable insights into the connection between network architecture and dynamics.

3:00–3:20 The *cd*-index of semi-Eulerian posets
Lorenzo Venturello (University of Siena, Italy)

Abstract. The numbers of flags of different ranks of an Eulerian poset are subject to a set of linear relations which have been described by Bayer and Billera. Fine showed that these relations are equivalent to the existence of a certain polynomial in two non-commutative variables, called the *cd*-index. In our work we show that it is possible to extend the definition of the *cd*-index from Eulerian to semi-Eulerian posets by a small modification of the flag f-polynomial. In particular we have the following.

Theorem 1. *Let P be a semi-Eulerian poset of rank $n+1$ with flag f-polynomial $\chi_P(a,b)$. The polynomial*

$$\chi'_P(a,b) := \chi_P(a,b) + (\chi(\mathbb{S}^{n-1}) - \chi(P))a\cdots ab$$

admits a cd-index $\Phi_P(c,d)$. Moreover, if P is simplicial we have that

$$\Phi_P(c,d) = \sum_{i=0}^{n-1} h_i(P)\check{\Phi}^n_i(c,d),$$

with $\check{\Phi}^n_i$ as defined by Stanley in [3] and $(h_0(P), h_1(P), \ldots h_n(P))$ is the h-vector of P.

Moreover, the coefficients of the *cd*-index of many interesting classes of posets (e.g., face posets of regular CW-spheres or Gorenstein* posets) are nonnegative, reflecting linear inequalities in the number of flags. Following partial results and a conjecture of Novik [2], we expect nonnegativity even for face posets of triangulated manifolds. In a joint work with Martina Juhnke and José Samper [1] we show that the coefficients of the *cd*-index of simplicial semi-Eulerian Buchsbaum posets are nonnegative. This family includes face posets of triangulated manifolds.

Theorem 2. *The coefficients of the cd-index of a semi-Eulerian Buchsbaum simplicial poset are nonnegative.*

This result is obtained by combining a lower bound proved by Novik and Swartz, together with a careful analysis of a family of non-commutative polynomials. This yields to stronger lower bounds on the coefficients than just nonnegativity. As a corollary we obtain nonnegativity of the so-called γ-polynomial of the order complex of Eulerian Buchsbaum simplicial poset of even rank. In particular, we underline the following result.

Corollary 3. *The order complex of any odd-dimensional triangulated manifold satisfies the Charney-Davis conjecture.*

References

[1] M. Juhnke, J. A. Samper, L. Venturello, *The cd-index of semi-Eulerian posets*, arXiv 2405.05812 (2024).

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[3] R. P. Stanley, *Flag f-vectors and the cd-index.* , Math. Z., 216:483–499, 1994.

3:30–3:50 Shellability of non ω-integral partition poset

Roberto Pagaria (Università di Bologna, Italy)

Abstract. Consider a graph Γ on the vertices $[r] = \{1, 2, \ldots, r\}$, a *flat* π is a flat of the associated graphical matroid. In other words, $\pi \vdash [r]$ is a set partition of the vertices such that for each block π_i of π the subgraph induced by π_i is connected. Let $\omega = (\omega_1, \ldots, \omega_r) \in \mathbb{R}^r$ be an integer vector.

Definition 1. A flat π is ω-integral if for each block π_i the sum $\sum_{j \in \pi_i} \omega_j \in \mathbb{Z}$ is an integer.

Let $\mathcal{L}_{\Gamma,\omega}$ be the poset of all non ω-integral flats of Γ ordered by refinement.

In general the poset $\mathcal{L}_{\Gamma,\omega}$ does not have a maximum, so we add a maximum element denoted by $\hat{1}$ and obtain the poset $\hat{\mathcal{L}}_{\Gamma,\omega} = \mathcal{L}_{\Gamma,\omega} \sqcup \{\hat{1}\}$.

Theorem 1. *The poset $\hat{\mathcal{L}}_{\Gamma,\omega}$ is LEX-shellable. In particular, its order complex is a wedge of spheres.*

The proof of this result is different from the proof of EL-shellability of partition posets and geometric lattices.

From this result we new obtain counting formula for lattice points in translated graphical zonotopes $Z_\Gamma + \omega$. As an application, we solve a problem in topology and representation theory: we determine the Ngo strings of the Hitchin fibration on the reduced locus.

References

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4:00–4:20 Tope-pair posets of oriented matroids and hyperplane arrangements

Emanuele Delucchi (University of Applied Arts and Sciences of Southern Switzerland, Switzerland)

Abstract. Oriented matroids encode the combinatorial structure of arrangements of pseudospheres, generalizing arrangements of hyperplanes in real vectorspaces. Salvetti complexes of oriented matroids represent a strictly wider class of homotopy types with respect to complements of hyperplane arrangements in complex vectorspaces (e.g., one obtains fundamental groups that cannot appear in the case of hyperplane arrangements).

To every real hyperplane arrangement and, more generally, to every oriented matroid, we associate a “tope-pair poset”. This poset is homotopy equivalent to the Salvetti complex and carries some extra structure that makes it a useful tool for combinatorial topology.

For instance, the tope-pair poset supports a free action of $\mathbb{Z}_4$ that discretizes the diagonal $\mathbb{C}^*$-action on complexified arrangement's complements, and it affords a proof of the fact that the integer cohomology of the Salvetti complex of any oriented matroid is given by the Orlik-Solomon algebra of the underlying matroid.

In the talk I will define the tope-pair poset, survey some of its features, and outline its connections to Artin groups and combinatorial fibrations.

The results presented in the talk are joint work with Michael Falk.

References

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5:00–5:20 $K(\pi, 1)$ - and other - conjectures for Artin groups
Mario Salvetti (Università di Pisa, Italy)

Abstract In the recent proof of the $K(\pi, 1)$-conjecture for affine Artin groups [1], well-established techniques from Discrete and Combinatorial Algebraic Topology, such as discrete Morse theory and shellability, play a fundamental role. We discuss (besides some aspects of the proof) related conjectures, particularly those stemming from the so-called *dual* structure (see also [2]), as well as explore some related combinatorial approaches.

References

[1] G. Paolini, M. Salvetti *Proof of the $K(\pi, 1)$-conjecture for affine Artin groups*, Inventiones Math.,vol. 224, p. 487-572, 2021.

[2] E. Delucchi, G. Paolini, M. Salvetti *Dual structures on Coxeter and Artin groups of rank three*, to appear in $\mathcal{G}\&\mathcal{T}$.

5:30–5:50 Vertex orders: From graphs to complexes
Bennet Goeckner (University of San Diego, USA)

Abstract. Many properties of graphs—such as Hamiltonicity and chordality—can be expressed in terms of orders on the vertices of the graph. Many of these definitions have been generalized to simplicial complexes (see [1]). We consider these properties in relation to the following type of shellings of simplicial complexes.

Definition 1. Let Δ be a pure simplicial complex with vertices labeled 1 through n. If the lexicographic order on the facets is a shelling of Δ, we say that Δ is *lex shellable*. Such an order on the vertices is a *lex shelling order*.

For many vertex orders stemming from graph properties, it is straightforward to determine their relationship (or lack thereof) with lex shelling orders. We focus primarily on *unit-interval orders*, for which the connection is more subtle.

Definition 2. Let Δ be a pure d-dimensional simplicial complex with vertices labeled 1 through n. Assume for any facet $F = v_0v_1 \dots v_d$ of Δ that the d-skeleton of $\{v_0, v_0 + 1, v_0 + 2, \dots, v_d\}$ is contained in Δ. Then Δ is a *unit-interval complex* and the order on the vertex set is a *unit-interval order*.

When $d = 1$, we recover a standard definition for unit-interval graphs G. That is, if $a < b < c$ are vertices of G and ac is an edge of G, then both ab and bc are edges of G. For complexes in general, we prove the following.

Theorem 1. *Let Δ be a pure and strongly connected simplicial complex. Then any unit-interval order for Δ is a lex shelling order.*

Time permitting, we will discuss similar vertex order properties and their connections to shelling completable complexes (see [2]).

References

[1] B. Benedetti, L. Seccia, M. Varbaro, *Hamiltonian paths, unit-interval complexes, and determinantal facet ideals*, Adv. in Appl. Math. 141 (2022), Paper No. 102407, 55 pp.

[2] M. Coleman, A. Dochtermann, N. Geist, S. Oh, *Completing and Extending Shellings of Vertex Decomposable Complexes*, SIAM J. Discrete Math. 36(2022), no.2, 1291–1305.

6:00-6:20 Chromatic Arrangements and Configuration Spaces with Obstacles

Sadok Kallel (American University of Sharjah, UAE, and Laboratoire Painlevé, Université de Lille, France)

Abstract. Given an abstract graph $\Gamma = (V, E)$, define the *chromatic configuration space* to be

$$\mathrm{Conf}_\Gamma(X) = \{(x_1, \cdots, x_n) \in X^{|V|} \mid x_i \neq x_j \text{ if } \{i, j\} \in E(\Gamma)\}$$

When the graph is complete, we get the classical configuration space of pairwise distinct points. We will show that $\mathrm{Conf}_\Gamma(X)$ splits, after only one suspension, into a bouquet of spheres, the number of spheres is computed precisely and is given in terms of some chromatic numbers of the graph. Our tools are poset topology and a geometric description of the homology classes. Similar ideas apply to a related kind of configuration spaces.

This work is joint with Moez Bouzouita (University of Tunis).

E–mail: antonio.rieser@cimat.mx.