AMS-UMI International
Joint Meeting 2024
Palermo, July 23–26, 2024

Advances in variational methods and applications
Special Session A26

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In the last decade, there has been a remarkable surge of interest within the scientific community, spanning physicists, engineers, biomedical researchers, and materials scientists, in the fields of new materials, micro-devices, artificial intelligence, machine learning, and stochastic modeling. These advancements have opened up a plethora of potential applications, driving the need for innovative mathematical techniques, methodologies, and the creation of novel function spaces to establish robust mathematical models. This requires a significant mathematical undertaking.

Our proposed minisymposium is dedicated to the variational formulation of these applications, aiming to provide well-founded and dependable mathematical descriptions. Our primary objective is to bridge the gap between these real-world applications and the realm of mathematics, uniting a community of experts in the field of Calculus of Variations.

This session is scheduled on July 23-24.

For more information visit `www.unipa.it`.

Schedule and Abstracts

July 23, 2024

11:00–11:30 Shape optimization for nonlocal anisotropic energies
Maria Giovanna Mora (University of Pavia, Italy)

Abstract Nonlocal shape optimization problems involving interaction energies with competing repulsive and attractive terms are of interest in a variety of applications and have been extensively studied in the last decades in the mathematical community. In this talk we will consider a family of nonlocal energies in 3d defined on sets with prescribed mass, where the repulsive interaction is an anisotropic variant of the Coulomb kernel and the attractive interaction is quadratic. Under the sole assumption of strict positivity of the Fourier transform of the anisotropic kernel, we will show that there is a critical value of the mass, above which ellipses are the unique minimizers and below which existence of minimizers fails. If instead the Fourier transform is just nonnegative, there is a dichotomy: either there is a critical mass as in the previous case or ellipses are minimizers for every mass. This behavior is related to the shape of minimizers when considering the energy on the larger class of measures with prescribed mass.

The talk is based on a joint project with Riccardo Cristoferi (Mathematics Department, Radboud University) and Lucia Scardia (Department of Mathematics, Heriot-Watt University)

11:30–12:00 Non-local perimeters from adversarial learning
Ryan Murray (Department of Mathematics, North Carolina State University)

Abstract Recent work in machine learning has recognized that many standard algorithms for classification are strongly affected by adversarial attacks. Accordingly, a growing body of research has tried to identify ways to mitigate this issue. This talk will discuss a natural non-parametric formulation of this objective, which can be transformed into a standard classification problem

that utilizes a non-local perimeter as a regularizer. Connections with optimal transportation, mean curvature flow, and minimal surfaces, and related open problems will also be discussed.

This represents joint work with Nicolás García Trillos, Leon Bungert, and Rachel Morris.

12:00–12:30 Geometric rigidity in variable domains and applications in dimension reduction

Leonard Kreutz (Department of Mathematics, Technical University of Munich)

Abstract In this talk we present a quantitative geometric rigidity estimate in dimensions $d = 2, 3$ generalising a celebrated result by Friesecke, James and Müller to the setting of variable domains. Loosely speaking, we show that for each function $y \in H^1(U; \mathbb{R}^3)$ and for each connected component of an open bounded set $U \subset \mathbb{R}^d$, the L^2-distance of ∇y from a single rotation can be controlled up to a constant by its L^2-distance from the group $SO(d)$, with the constant not depending on the precise shape of U, but only on an integral curvature functional related to ∂U. We further show that for linear strains the estimate can be refined, leading to a uniform control independent of the set U. The estimate can be used to establish compactness in the space of generalized special functions of bounded deformation (GSBD) for sequences of displacements related to deformations with uniformly bounded elastic energy. We show how this estimate can be applied in the context of dimension reduction by calculating the Γ-limit of a model for thin elastic solids containing voids.

The talk is based on a joint project with Manuel Friedrich (Department of Mathematics, University of Erlangen) ans Konstantinos Zemas (Department of Mathematics, University of Bonn)

12:30–13:00 A variational model for 3D features in films/foams

Michael Novack (Department of Mathematical Science, Carnegie Mellon University)

Abstract Area minimization among a suitable class of 2D surfaces is the basic variational model describing the interfaces in films/foams. In this talk we will discuss a modification of this paradigm in which surfaces are replaced with regions of small but positive volume. The model captures features of real films/foams, such as Plateau borders, that cannot be described by zero volume objects. We will also discuss the PDE approximation of this problem via the Allen-Cahn equation and its relation to Plateau's laws, which govern the possible singularities.

The talk is based on a joint porject with Francesco Maggi (Department of Mathematics, University of Texas at Austin) and Daniel Restrepo (Department of Mathematics, Johns Hopkins University)

14:30-15:00 Stochastic homogenization in micromagnetics

Lorenza D'Elia (Institute of Analysis and Scientific Computing, TU Wien)

Abstract Many key properties and applications of magnetic materials are strongly intertwined with the spatial distribution of magnetic moments inside the corresponding specimens. In addition to classical magnetic structures, magnetic skyrmions have raised interest in spintronics as carriers of information for future storage devices. The chirality of magnetic skyrmions is determined by the so-called Dzyaloshinkskii-Moriya (DM) interaction.

In this talk, we present an advance in the mathematical modeling of magnetic skyrmions by analyzing the interplay of stochastic microstructures and chirality. Under the assumptions of stationarity and ergodicity, we characterize the Gamma-limit of a micromagnetic energy functional, including the DM contribution. Eventually, we present an explicit characterization of minimizers of the effective model in the case of magnetic multilayers.

This talk is based on a joint work with E. Davoli (Institute of Analysis and Scientific Computing, TU Wien) and J. Ingmanns (Institute of Science and Technology Austria).

15:00- 15:30 Sharp-interface limit for non-isothermal and nonlocal Modica-Mortola functionals.

Emanuele Tasso (Institute of Analysis and Scientific Computing, TU Wien) [1]

Abstract In this talk, we analyze a non-isothermal and nonlocal variant of the Modica-Mortola diffuse model for phase transitions fo the form

$$F_\epsilon(u;\Omega) := \frac{1}{4\epsilon}\int_{\Omega\times\Omega} J_\epsilon(x'-x)|u(x')-u(x)|^2\,dx'dx + \frac{1}{\epsilon}\int_\Omega W(x,u(x))\,dx.$$

Here the classical gradient penalization is replaced by a nonlocal singular perturbation and the double-well potential is space-dependent. Our main result is the identification of the sharp-interface limit as the width ϵ of the transition layers converges to zero. This is joint work with Elisa Davoli (Institute of Analysis and Scientific Computing, TU Wien).

15:30-16:00 Regularity for minimizers of the Griffith fracture energy

Kerrek H. Stinson (Hausdorff Center for Mathematics, University of Bonn)

Abstract The Griffith criterion says that the energy to crack a brittle elastic material is proportional to the length of the crack. Understanding the regularity of minimizers requires unraveling the complex interplay of bulk (elastic) and surface (crack) energies in the vectorial setting of linearized elasticity. In dimension 2, we prove that the crack of a minimizer is given by a $C^{1,1/2}$ surface outside of a singular set of points with dimension strictly less than 1, analogous to results for the scalar-valued Mumford-Shah functional.

The talk is based on a joint project with Manuel Friedrich (Department of Mathematics, Friedrich-Alexander-Universität Erlangen-Nürnberg) and Camille Labourie (Department of Mathematics, Université Paris-Saclay).

16:00-16:30 Regularity and compactness for critical points of degenerate polyconvex energies

André Guerra (ETH Zürich)

Abstract Let $\Omega\subset\mathbb{R}^2$ be a smooth, bounded domain and consider the polyconvex energy

$$\mathbb{E}[u] \equiv \int_\Omega g(\det \mathrm{D}u)\,\mathrm{d}x, \qquad u\colon\Omega\to\mathbb{R}^2,$$

where g is a C^1 strictly convex function. Energies of this type were studied extensively in the literature, for instance in connection with elastic fluids or with the prescribed Jacobian equation.

In [1] we consider critical points of $\mathbb{E}$, i.e. solutions of the Euler–Lagrange system

$$\operatorname{div}(g'(\det \mathrm{D}u)\operatorname{cof}(\mathrm{D}u)) = 0. \tag{1}$$

Any C^1 solution of (1) satisfies $\det \mathrm{D}u = c$, for some $c\in\mathbb{R}$. Our first result is that the same rigidity continues to hold for Lipschitz solutions:

Theorem 1. *Suppose that u is a Lipschitz solution of (1), where $g\in C^1$ is strictly convex. Then $\det \mathrm{D}u$ is constant a.e. in Ω.*

The proof of this theorem combines ideas from the DiPerna–Lions theory for renormalized solutions of the continuity equation, and methods from quasiconformal analysis.

We then investigate the compactness properties sequences of approximate solutions to (1), i.e. sequences such that

$$\operatorname{div}(g'(\det \mathrm{D}u_j)\operatorname{cof}(\mathrm{D}u_j)) = \operatorname{div}(F_j), \qquad F_j\to 0 \text{ in } L^1(\Omega). \tag{2}$$

This question is closely related to the stability of (1). We show that the Jacobians of approximate solutions are in fact pre-compact:

Theorem 2. *Let (u_j) be a bounded sequence of Lipschitz maps satisfying (2). Then, up to a subsequence, u_j converges weakly-$*$ in $W^{1,\infty}$ to a solution of (1), and $\det \mathrm{D}u_j$ converges to $\det \mathrm{D}u$ strongly in L^p for any $p<\infty$.*

[1] This work was partially funded by the Austrian Science Fund (FWF)

In particular, we obtain the following, which answers positively [2, Question 10]:

Corollary 3. *The differential inclusion associated with* (1) *is quasiconvex.*

The paper represents joint work with R. Tione (MPI Leipzig).

References

[1] A. Guerra, R. Tione (2024). Regularity and compactness for critical points of degenerate polyconvex energies. arXiv:2401.16315.

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July 24, 2024

11:30-12:00 Discrete-to-continuous crystalline curvature flows

Massimiliano Morini (University of Parma)

Abstract We consider a fully discrete variant of the implicit variational scheme for mean curvature flow, in a setting where the flow is governed by a crystalline surface tension defined by the limit of pairwise interactions energy on the discrete grid. The algorithm is based on a new discrete distance from the evolving sets, which prevents the occurrence of the spatial drift and pinning phenomena that usually appear in similar discrete frameworks. We provide the first rigorous convergence result holding in any dimension, for any initial set and for a large class of purely crystalline anisotropies, in which the spatial discretization mesh can be of the same order or even coarser than the time step.

12:00 – 12:30 Existence of minimizers for a two-phase free boundary problem with coherent and incoherent interfaces

Paolo Piovano (Dipartimento di Matematica, Politecnico di Milano)[2]

Abstract A variational model for describing the morphology of two-phase continua by allowing for the interplay between coherent and incoherent interfaces is discussed. Coherent interfaces are characterized by the microscopical arrangement of atoms of the two materials in a homogeneous lattice, with deformation being the solely stress relief mechanism, while at incoherent interfaces delamination between the two materials occurs. The model is designed in the framework of the theory of Stress Driven Rearrangement Instabilities, which are characterized by the competition between elastic and surface effects. The existence of energy minimizers is established in the plane by means of the direct method of the calculus of variations under a constraint on the number of boundary connected components of the underlying phase, whose exterior boundary is prescribed to satify a graph assumption, and of the two-phase composite region. Both the wetting and the dewetting regimes are included in the analysis.

This talk is based on a joint project with Randy Llerena (Faculty of Mathematics, University of Vienna).

12:30 – 13:00 Existence of solutions past collisions for viscoelastic solids

Giovanni Gravina (School of Mathematics and Statistical Sciences, Arizona State University)

Abstract In this talk, we will examine the time evolution of viscoelastic solids within a framework that allows for collisions and self-contact. In the static and quasi-static regimes, corresponding existence results have been shown through variational descriptions of the problem. For the fully dynamical case, however, collisions have so far either been ignored or handled using a simplified model (for example, using repulsive terms). In contrast to this, by employing a newly developed variational technique, we are able to prove the existence of solutions for arbitrary times. This entails the study of the measure-valued contact forces that naturally emerge when enforcing non-interpenetration of matter.

The talk is based on a joint work with Antonín Češík (Department of Mathematical Analysis Faculty of Mathematics and Physics, Charles University, Prague, Czech Republic) and Malte

[2]Aknowledgements: Projects P 29681 and TAI 293 by the Austrian Science Fund (FWF), Project MA16-005 by the Vienna Science and Technology Fund (WWTF), Projects CUP: E55F22000270001 and E53C22001930001 by GNAMPA-INdAM.

Kampschulte (Department of Mathematical Analysis, Faculty of Mathematics and Physics, Charles University, Prague, Czech Republic).

14:30 – 15:00 Global existence and uniqueness of a micro-macro model for reactive transport in elastic perforated media

Maria Neuss-Radu (Department of Mathematics, Friedrich-Alexander-Universität Erlangen-Nürnberg)[3]

Abstract We consider an effective model of micro-macro type consisting of a macroscopic elasticity-transport problem and associated cell problems, formulated in a unified *Lagrangian* framework. This model has been derived by a formal asymptotic expansion from a microscopic model defined in an elastically deformable perforated medium and formulated in a mixed *Eulerian/Lagrangian* framework. The effective model is nonlinearly coupled through reaction terms as well as effective coefficients which take into account the periodic microstructure and, in the case of the transport problem, the deformation of the domain. We prove global existence and uniqueness of a weak solution under a smallness assumption on the data of the elasticity subsystem. This talks is based on a joint project with Jonas Knoch (Department of Mathematics, Friedrich-Alexander-Universität Erlangen-Nürnberg) and Markus Gahn (Institute for Mathematics, University Heidelberg).

References

[1] J. Knoch, M. Gahn, M. Neuss-Radu *Global well-posedness and numerical justification of an effective micro-macro model for reactive transport in elastic perforated media*, arXiv:2404.04664

15:00 – 15:30 Rigorous derivation of effective models for perforated second grade viscoelastic materials

Markus Gahn (Institute for Mathematics, University Heidelberg)

Abstract In this talk, we perform the homogenization for nonlinear viscoelastic non-simple perforated materials at large strain in the quasistatic setting. Starting from a microscopic model on a periodically perforated reference domain depending on a scaling parameter ε giving the ratio between the size of the whole domain and the small periodic perforations, for $\varepsilon \to 0$ we rigorously derive a macroscopic model including homogenized coefficients. The mechanical energy in the micro-model depends on both, the gradient and the second gradient of the deformation, and also respects positivity of the determinant of the deformation gradient. Further, we assume dynamic frame indifference for the viscous stresses. For the homogenization we use the two-scale convergence and the unfolding method. The most crucial step is to establish ε-uniform a priori estimates for the microscopic solutions, in particular for the rate of the deformation gradient. For this we construct new extension operators for second order Sobolev spaces and prove a Korn inequality for non-constant coefficients on the perforated domain. Another crucial aspect is to guarantee an ε-independent positive lower bound for the determinant of the deformation gradient.

15:30 – 16:00 Nematic soap films

Luca Lussardi (Dipartimento di Scienze Matematiche "G.L. Lagrange", Politecnico di Torino)

Abstract Nematic films are thin fluid structures, ideally two dimensional, endowed with an in-plane degenerate nematic order. Some variational models for nematic films have been introduced by Giomi in 2012 and by Napoli and Vergori in 2018. At equilibrium, the shape of the nematic film results from the competition between surface tension, which favors the minimization of the area, and the nematic elasticity, which instead promotes the alignment of the molecules along a common direction. The main difference between the two mentioned approaches is the way to compute the surface derivative of the nematic vector field. In this seminar I will briefly describe the models and I will present some recent analytical results obtained in collaboration with Giulia Bevilacqua, Chiara Lonati and Alfredo Marzocchi.

[3] Acknowledgemente: Graduiertenkolleg 2339 IntComSin of the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) – Project-ID 321821685.

16:00 – 16:30 Calderòn Problem for Nonlinear Electrical Conductivity

Luisa Faella (Department Electrical and Information Engineering "Maurizio Scarano", University of Cassino and Southern Lazio)

We treat an inverse electrical conductivity problem, which deals with the reconstruction of nonlinear electrical conductivity starting from boundary measurements in steady currents operations. In this framework, a key role is played by the Monotonicity Principle, which establishes a monotonic relation, connecting the unknown material property to the (measured) Dirichlet-to-Neumann operator (DtN). We prove that the Monotonicity Principle for the Dirichlet energy in nonlinear problems holds under mild assumptions. Then, we show that apart from linear and p-Laplacian cases, transferring this monotonicity result from the Dirichlet energy to the DtN operator is impossible. To overcome this issue, we introduce a new boundary operator, identified as the average DtN operator. Moreover, the nonlinear constitutive relationships, at a given point in the space, present a behavior for large arguments described by monomials of order p and q so that a weighted $p-$Laplacian problem can approximate the nonlinear problem.

References

[1] A. Tamburrino and G. Rubinacci, *A new non-iterative inversion method for electrical resistance tomography*, Inverse Problems 18, 1809–1829, 2002.

[2] A. Tamburrino and G. Rubinacci, *Fast methods for quantitative eddy-current tomography of conductive materials*, IEEE Transactions on Magnetics, 42,2017–2028, 2006.

[3] A.C. Esposito, L Faella, G Piscitelli, R Prakash, A Tamburrino, *Monotonicity principle in tomography of nonlinear conducting materials*, Inverse Problems 37 (4), 045012, 2021.

[4] A.C. Esposito, L Faella, G Piscitelli, V Mottola, R Prakash, A Tamburrino, *The p_0 -Laplace "Signature" for Quasilinear Inverse Problems*, SIAM Journal on Imaging Sciences 17 (1), 351-388, 2024.

[5] A.C. Esposito, L Faella, G Piscitelli, R Prakash, A Tamburrino, *The p–Laplace "Signature" for Quasilinear Inverse Problems with Large Boundary Data*, SIAM Journal on Mathematical Analysis 56 (1), 275-303, 2024.

17:00 - 17:30 Structure of the boundary singular set of area minimizing currents

Reinaldo Resende (Department of Mathematics, Carnegie Mellon University.)

Abstract Let T be an area minimizing integral m-current in $\mathbb{R}^{m+n}$. It is well known that regularity theory in geometric measure theory is trickier when considering codimensions higher than 1. Almgren wrote his whole treatise [3] to prove that the Hausdorff dimension of the interior singular set of T cannot exceed $m-2$.

The boundary regularity for an area minimizing integral m-current in $\mathbb{R}^{m+n}$ taking the $C^{3,\alpha}$ boundary Γ with multiplicity Q is known to be weaker. Indeed, the following example given in [2] evidenciate it.

Proposition 1 *There exists a* 2*-dimensional area minimizing integral current T in $\mathbb{R}^4$ taking its boundary $\Gamma \in C^\infty$ (which is given by two disjoint closed curves) with multiplicity* 1 *and such that the Hausdorff dimension of the boundary singular set is* 1.

In [1], we are able to prove the following theorem.

Theorem 2 *Let T be an area minimizing integral m-current in $\mathbb{R}^{m+n}$ taking the boundary with multiplicity $Q \in \mathbb{N} \setminus \{0\}$. Then the boundary regular set of T is open and dense.*

Moreover, denoting $\mathcal{H}^{m-3}$ as the $(m-3)$-dimensional Hausdorff measure in $\mathbb{R}^{m+n}$, we can give further information about the structure of the set of one-sided boundary singular points. We call p an one-sided boundary point of T, if the density of T at p equals to half of the multiplicity of the boundary, i.e., $\Theta^m(T,p) = Q/2$.

Theorem 3 *Let T be an area minimizing integral m-current in $\mathbb{R}^{m+n}$ taking the boundary with multiplicity $Q \in \mathbb{N} \setminus \{0\}$. Then the set of one-sided singular points is $\mathcal{H}^{m-3}$-rectifiable.*

References

[1] I. Fleschler, R. Resende. Boundary regularity of area minimizing m-currents with arbitrary multiplicity, to appear in arXiv.

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17:30 – 18:00 Asymptotic analysis of thin structures with point-dependent energy growth

Michela Eleuteri (Department of Physics, Informatics and Mathematics, University of Modena and Reggio Emilia)

Abstract $3D - 2D$ dimensional reduction for hyperelastic thin films modelled through energies with point-dependent growth, assuming that the sample is clamped on the lateral boundary, is performed in the framework of Γ-convergence. Integral representation results, with a more regular Lagrangian related to the original energy density, are provided for the lower dimensional limiting energy, in different contexts.

This talk is based on a joint work with Francesca Prinari (Department of Agricultural, Food and Agro-environmental Sciences, University of Pisa) and Elvira Zappale (Department of Basic and Applied Sciences for Engineering, Sapienza - University of Rome)

18:00–18:30 Dynamic brittle fracture formulated as an initial value problem

Robert Lipton (Louisiana State University, USA)[4]

Abstract A nonlocal model for dynamic brittle fracture is introduced consisting of two phases, one elastic and the other inelastic. Evolution from the elastic to the inelastic phase depends on material strength. Existence and uniqueness of the displacement-failure pair follow from the initial value problem. The displacement-failure pair satisfies energy balance. The length of nonlocality ϵ is taken to be small relative to the domain. The evolution provides an energy that interpolates between volume energy corresponding to elastic behavior and surface energy corresponding to failure. In general the deformation energy resulting in material failure over a region is a bounded $d-1$ dimensional integral. For fixed $\epsilon > 0$, the failure energy is nonzero for $d-1$ dimensional regions associated with flat crack surfaces. This failure energy is the Griffith fracture energy given by the energy release rate multiplied by area of the crack. The nonlocal field theory is shown to recover a solution of the linear elastic wave equation equation outside a propagating flat traction free crack in the limit of vanishing spatial nonlocality. For curved or more generally countably rectifiable cracks the failure energy is the Griffith fracture energy but only in the $\epsilon \to 0$ limit. Weak convergence methods, slicing variables and methods of geometric measure theory are used to demonstrate claims. This is part of a joint work with D. Bhattacharya.

References

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[4]This material is based upon work supported by the U.S. Army Research Laboratory and the U. S. Army Research Office under Contract/Grant Number W911NF-19-1-0245.

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