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Computability Theory
Special Session A14

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Computability is a fundamental notion of mathematics. Interest in effectiveness is already apparent in the famous Hilbert problems, in particular, the second and tenth, and in early 20th century work of Dehn, initiating the study of word problems in group theory. The formal definition of computability was given by Turing, Gödel, and others in the 1930s. Problems are classified according to various logical hierarchies, giving precise complexity measurements, which closely relate computational and definitional complexity.

Since its inception, and perhaps especially so in the past 30–40 years, computability theory has seen many fascinating and dramatic developments, growing to encompass many new subfields. These include:

- Algorithmic randomness: The study of randomness for individual objects such as reals.
- Computable structure theory: The study of computational aspects of mathematical structures.
- Reverse mathematics: The search for optimal axioms to prove mathematical theorems.
- Reducibility and degree structures: The study of the Turing degrees, and more generally of various notions of comparison of computability-theoretic strength.

Likewise, the subject has found links to other mathematical disciplines inside and outside of logic. This includes, for instance, work on enumeration degrees that has revealed deep and surprising relations to general topology, and work on algorithmic randomness that is closely tied to symbolic dynamics and geometric measure theory. Inside logic there are relations to model theory, set theory, effective descriptive set theory, and proof theory.

In some of these cases the bridges to seemingly distant mathematical fields have yielded completely new proofs or even solutions of open problems in the respective fields. In others, previously disparate areas have found common tools and questions, resulting in what are now essentially merged fields. An example of the latter is reverse mathematics and computable analysis, which have become deeply intertwined through new developments over the past decade.

The special session will cover the majority of areas of modern computability theory, including all those mentioned above.

For more information visit `www.computability.org/umi-ams`.

Schedule and Abstracts

July 23, 2024

11:00–11:45 Minimal covers in the Weihrauch degrees
Manlio Valenti (University of Swansea, UK)

Abstract. Weihrauch reducibility is a notion of reducibility that calibrates the uniform computational strength of computational problems. Despite its growing popularity, the structure of Weihrauch degrees is vastly unexplored, as most of the efforts up to this date have concentrated on the classification of the Weihrauch degrees of specific problems.

In this talk, after a short introduction to the topic we will explore some recent developments on the structure of Weihrauch degrees. Recall that, given a partial order $(P, \leq)$, we say b is a minimal cover of a if $a < b$ and there is no c such that $a < c < b$. In other words, b is a minimal cover of a if the interval (a,b) is empty. We say that b is a strong minimal cover of a if $c<b$ implies $c \leq a$.

We present a complete characterization of minimal covers and strong minimal covers in the Weihrauch degrees. Let $\{p\}^+ = \{\langle e,q\rangle : \Phi_e(p)=q \text{ and } q \not\leq_{\mathrm{T}} p\}$.

Theorem 1. *Let f and h be partial multi-valued functions on Baire space. The following are equivalent:*

(1) *f is a minimal cover of h in the Weihrauch degrees.*
(2) *$f \equiv_{\mathrm{W}} h \sqcup \mathrm{id}_{\{p\}}$ for some p with $\mathrm{dom}(h) \not\leq_{\mathrm{M}} \{p\}$ and $\mathrm{dom}(h) \leq_{\mathrm{M}} \{p\}^+$.*

Theorem 2. *Let f and h be partial multi-valued functions on Baire space. The following are equivalent:*

(1) *f is a strong minimal cover of h in the Weihrauch degrees.*
(2) *There is $p \in \mathbb{N}^{\mathbb{N}}$ such that $f \equiv_{\mathrm{W}} \mathrm{id}_{\{p\}}$ and $h \equiv_{\mathrm{W}} \mathrm{id}_{\{p\}^+}$.*

The previous two results have interesting consequences: they imply that the degree of id is first-order definable in $(\mathcal{W}, \leq_{\mathrm{W}})$ and that the first-order theory of the Weihrauch degrees (below id) is recursively isomorphic to the third-order theory of true arithmetic.

This is joint work with Steffen Lempp, Joe Miller, Arno Pauly, and Mariya Soskova.

12:00–12:20 Computability theory and existentially closed groups
Isabella Scott (University of Chicago, USA)

Abstract. Existentially closed groups were introduced in 1951 in analogue with algebraically closed fields. Since then, they have been further studied by Neumann, Macintyre, and Ziegler, who elucidated deep connections with model theory and computability theory. We review some of the literature on existentially closed groups and present new results that further refine these connections.

References

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12:30–12:50 Characterizing learnability for families of structures
Vittorio Cipriani (Technische Universität Wien, Austria)

Abstract. In this talk, we study learning of countable families of countable relational structures. The framework we use combines ideas from computable structure theory and inductive inference, and it was defined in a series of papers by Bazhenov, Fokina, Kötzing, and San Mauro ([2],[3]). The framework models the scenario in which, given a family of structures $\mathfrak{K}$, a *learner* receives more and more information about the atomic diagram of a copy of some $\mathcal{A} \in \mathfrak{K}$ and, at each stage, is required to output a conjecture about the isomorphism type of such a structure.

A natural criterion to consider is **Ex**-*learning* in which we require the learner to stabilize to the correct conjecture in finitely many steps. Together with Bazhenov and San Mauro in [1] we gave a descriptive set-theoretic characterization of **Ex**-learning. Namely, we showed that a family of structures is **Ex**-learnable if and only if the corresponding isomorphism problem continuously reduces to E_0, the equivalence relation of eventual agreement on $2^{\mathbb{N}}$. Replacing E_0 with other equivalence relations, one obtains a hierarchy to rank such isomorphism problems. That is, a family of structures $\mathfrak{K}$ is *E-learnable* for an equivalence relation E if there is a continuous reduction from the isomorphism problem associated with $\mathfrak{K}$ to E.

To get a better understanding of which families of structures are E-learnable it is useful to provide a model-theoretic characterization of E-learnability. This has been done for some equivalence relations (see e.g., [2, 1]): these characterizations rely on the ordering between the structures in the family with respect to the inclusion of their Σ_2^{inf}-theories. A natural question is to ask for every n which is (if exists) the equivalence relation E^n such that a family of structures $\mathfrak{K}$ is E^n-learnable if and only if the ordering between the structures in $\mathfrak{K}$ with respect to the

inclusion of their $\Sigma_n^{\inf}$-theories is a partial order. It turns out that these equivalence relations exist and they are natural ones: namely, they are the (iterations of the) Friedman-Stanley jump of equality on $\mathbb{N}$ and $2^{\mathbb{N}}$.

We continue showing that other learning criteria coming from the classical setting of inductive inference of formal languages or recursive functions have nice model-theoretic characterization as well.

This talk collects joint works with Bazhenov, Jain, Marcone, San Mauro and Stephan.

References

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[2] Nikolay Bazhenov, Ekaterina Fokina, and Luca San Mauro. Learning families of algebraic structures from informant. *Information and Computation*, 275:104590, 2020.

[3] Ekaterina Fokina, Timo Kötzing, and Luca San Mauro. Limit learning equivalence structures. In Aurélien Garivier and Satyen Kale, editors, *Proceedings of the 30th International Conference on Algorithmic Learning Theory*, volume 98 of *Proceedings of Machine Learning Research*, pages 383–403, Chicago, Illinois, 22–24 Mar 2019. PMLR.

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14:30–15:15 On structures with non-computable presentations
Ekaterina Fokina (Technische Universität Wien, Austria)

Abstract. In computable structure theory, one usually measures the complexity of a structure by identifying the structure with its atomic diagram. The structure is then computable if so is its atomic diagram. The notion naturally relativises to non-computable oracles. However, for structures without computable presentations, a finer way to measure the complexity is sometimes more natural and suitable.

In this talk we discuss different approaches to measure the complexity of non-computable structures and explain several of our recent results.

15:30–15:50 On normality, supernormality, finite state dimension, and point to set principles
Elvira Mayordomo (Universidad de Zaragoza, Spain/Iowa State University, USA)

Abstract. Effective and resource-bounded dimensions were defined by Lutz in [6] and [5] and have proven to be useful and meaningful for quantitative analysis in the contexts of algorithmic randomness, computational complexity and fractal geometry (see the surveys [2,7,3,10] and all the references in them).

The point-to-set principle (PSP) of J. Lutz and N. Lutz [8] fully characterizes Hausdorff and packing dimensions in terms of effective dimensions in the Euclidean space, enabling effective dimensions to be used to answer open questions about fractal geometry, with already an interesting list of geometric measure theory results (see [4,9]).

Finite state dimension [1] is the lowest level effectivization of Hausdorff dimension and is closely related to Borel normality. In this talk I will review its main properties, prove a new characterization in terms of information content approximated at a certain precision, and consider new generalizations of normality. I will then prove a finite-state dimension point to set principle [11].

References

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[9] J. H. Lutz and E. Mayordomo. Algorithmic fractal dimensions in geometric measure theory. In V. Brattka and P. Hertling, editors, *Handbook of Computability and Complexity in Analysis*. Springer-Verlag, 2021.

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[11] E. Mayordomo. A point to set principle for finite-state dimension. Technical Report arXiv:2208.00157, Arxiv, 2022.

16:00–16:20 Nash-Williams theorem for sequences of finite range in ATR_0

Giovanni Soldà (Universiteit Gent, Belgium)

Abstract. In [1], Crispin Nash-Williams proved that if $(Q, \leq_Q)$ is a *well quasi-order* (henceforth *wqo*), then so is the set of transfinite sequences over Q with finite range, ordered by embeddability. In this talk, we will show that this result is provable in the subsystem of second-order arithmetic ATR_0: together with previous results by Shore [2], this determines the reverse mathematical strength of Nash-Williams' result.

We will see that, interestingly, the proof relies on the notion of *better quasi-order* (henceforth *bqo*), a strengthening of the concept of wqo again due to Nash-Williams. We will present how the "degree of bqo-ness" of a quasi-order Q relates to the order-theoretic properties of $Q^{<\omega^{1+\alpha}}$, the set of sequences over Q of length less than $\omega^{1+\alpha}$, and to those of $\dot{V}_\alpha(Q)$, which corresponds (roughly) to the α-iterated powerset of Q. These observations will lead to an equivalence between $\dot{V}_\alpha(Q)$ and the *indecomposable* sequences in $Q^{<\omega^{1+\alpha}}$, and, together with a slight modification of clopen Ramsey theorem, will lead to the desired proof in ATR_0.

Joint work with Fedor Pakhomov.

References

[1] C. Nash-Williams, *On well-quasi-ordering transfinite sequences*, Mathematical Proceedings of the Cambridge Philosophical Society 61 (1965), no. 1, 33–39.

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17:00–17:45 From combinatorial theorems to well-ordering principles

Lorenzo Carlucci (Università di Roma "La Sapienza", Italy)

Abstract. Many well-ordering preservation principles and theorems from Ramsey Theory are known to be equivalent to various systems of Reverse Mathematics. For example Ramsey's Theorem for triples and the well-ordering preservation principle for base-ω exponentiation are both equivalent to Arithmetical Comprehension. The resulting equivalence between the combinatorial theorem and the well-ordering principle is usually obtained indirectly.

We present some examples of direct implications from Ramsey-type and Hindman-type theorems to well-ordering principles. The proofs yield computable reductions. For example we show

that the well-ordering principle for the ε-ordering reduces to Ramsey's Theorem for colorings of exactly large sets. This work is joint with Mainardi and Zdanowski.

We furthermore present some preliminary results on generalizations of the Free Set, Thin Set and Rainbow Ramsey Theorem to colorings of exactly large sets (and, more generally, to colorings of barriers) for which we obtain weak anti-basis results. This work is joint with Gjetaj and Vivi.

July 24, 2024

11:30–11:50 Weihrauch degrees of indivisibility

Arno Pauly (University of Swansea, UK)

Abstract. We call a structure $\mathcal{M}$ over $\mathbb{N}$ *indivisible*, if for every colouring of $\mathbb{N}$ with finitely many colours there is a monochromatic isomorphic copy of $\mathcal{M}$. A typical example of an indivisible structure is $(\mathbb{Q}, <)$. For a fixed indivisible structure $\mathcal{M}$ we can then study the computational task $\mathrm{Ind}\mathcal{M}$, which receives as input a k-colouring of $\mathbb{N}$ and which has to output a monochromatic copy of $\mathcal{M}$. This programme was recently formulated by Kenneth Gill [3, 4], who obtained results about the Weihrauch degree [1] of $\mathrm{Ind}\mathbb{Q}$ and some other structures. In a largely independent development, Dzhafarov, Solomon and Valenti [2] studied the Weihrauch degree of the tree pigeon hole principle TT^1_+, which can be seen as indivisibility of the full binary tree with relations for "is in the left subtree below" and "is in the right subtree below". It is easy to see that $\mathrm{Ind}\mathbb{Q} \equiv_{\mathrm{W}} \mathrm{TT}^1_+$.

Our contribution here is to solve some of the questions left open in these two projects. In particular, we prove that:

Theorem 3. $\mathrm{C}_{k+1} \not\leq_{\mathrm{W}} \mathrm{TT}^1_k$ *for all* $k \geq 1$.

Here C_{k+1} is *finite closed choice* with $k+1$ elements, the problem receives as input an enumeration of some (maybe none) but not all elements of $\{0, 1, \ldots, k-1\}$, and the valid outputs are the elements not appearing in the input. With TT^1_k we denote the restriction of the tree pigeon hole principle to k colours. Overall the gist of the resulting picture is that the opportunities to code more discrete information into an instance of TT^1_+ (or equivalently $\mathrm{Ind}\mathbb{Q}$) beyond which colours can appear in solutions are very limited (but exist).

References

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[3] Kenneth Gill. *Two studies in complexity.* PhD thesis, Pennsylvania State University, 2023.

[4] Kenneth Gill. Indivisibility and uniform computational strength. arXiv 2312.03919, 2024.

12:00–12:20 What can be uniformly computed from descending sequences?

Jun Le Goh (National University of Singapore, Singapore)

Abstract. How hard is it to find an infinite descending sequence in a countable ill-founded linear order? What about an infinite bad sequence in a countable non-well-quasi-order? We investigate these questions in the computability-theoretic framework of Weihrauch (uniform) reducibility, where the above problems are denoted DS and BS respectively. First, we will show that DS cannot compute an infinite path in a given finitely branching tree, resolving the primary open question in our previous paper [1]. Then we will show that DS cannot compute even a single element which is extendible (to an infinite bad sequence) in a given non-well-quasi-order, refuting our false claim in [1]. This talk is based on ongoing joint work with Arno Pauly and Manlio Valenti [2].

References

[1] J. L. Goh, A. Pauly, M. Valenti, *Finding descending sequences through ill-founded linear orders*, The Journal of Symbolic Logic, Volume 86, Issue 2, June 2021, pp. 817–854.

[2] J. L. Goh, A. Pauly, M. Valenti, *The weakness of finding descending sequences in ill-founded linear orders*, arXiv:2401.11807, 2024.

12:30–12:50 Computability of the Whitney Extension Theorem

Guido Gherardi[1] (Università di Bologna, Italy)

Abstract. A very relevant topic in computable analysis which, not surprisingly, has marked the development of the subject is the *effective* reformulation of classical mathematical theorems of the form

$$(\forall x \in X)(\exists y \in Y)A(x, y) \quad (*)$$

for X and Y suitable topological spaces. This exactly corresponds to showing that a corresponding multi-valued function, assigning to any $x \in X$ a $y \in Y$ such that $A(x, y)$, is *computable*.

A strictly related interesting aspect, also from a philosophical perspective, is to evaluate to what extent the classical proofs of those classical results have an algorithmic nature. Some of such proofs present indeed an intuitive computational flavour, but, so to say, some computable steps are hidden under the surface. Hence, in such cases, the proofs of computability of the corresponding multi-valued functions can retrace classical proofs by showing in a rigorous way the computational content that was only sketched in them. Nevertheless, removing the rind to get to the computational pith might contain non-trivial steps, and it depends first of all on the correct choice of suitable translations of the classical concepts into computational notions.

An interesting example is given by the well-known Urysohn-Tietze Extension Theorem. A computable version of this theorem was proved by Klaus Weihrauch in [4]. In order to prove this result, Weihrauch provided in fact an *effectivization* of the classical proof contained in [1].

The Urysohn-Tietze Theorem classically finds a generalization in the Whitney Extension Theorem. For the real case, this theorem states that for any given (non-empty) closed set $F \subseteq \mathbb{R}^n$ and a *jet* of order m of functions on F, there exists a total continuous function g in $C_m(\mathbb{R}^n)$ such that both g as well as its partial derivatives coincide on F with the corresponding partial functions of the jet. Here a jet is a finite sequence of continuous functions defined on F satisfying Taylor's condition, and which behave like partial derivatives of each other. A classical proof of this statement is contained in [3] and, as a preliminary result, Stein proves an extension theorem for the limit case in which the jet consists only of a single continuous function, providing then another proof for the Urysohn-Tietze Extension Theorem.

In this talk I will check the computability of the construction of Whitney's extensions through an effectivization of the proof given by Stein. A preliminary investigation of its computational content brought already in [2] to the systematic classification of different formulations of the projection point problem onto closed subsets of $\mathbb{R}^n$, and it turned out that only the problem of finding approximated projection points of an $x \in \mathbb{R}^n$ onto a closed $F \subseteq \mathbb{R}^n$ up to a given error bound ε is computable with respect to full information for closed sets (which consists of an exact open covering of the complement of the set and of a dense subset of the set itself). In fact, the effectivization of Stein's proof requires the use of approximations in several different aspects and in a way that implies a non-trivial departure from its original formulation.

Joint work with Andrea Brun and Alberto Marcone.

References

[1] R. Engelking, *General Topology*, 1989.

[2] G. Gherardi, A. Marcone, A. Pauly, *Projection operators in computable analysis*, Computability 8 (2019) 281–304.

[3] E. Stein, *Singular integrals and differentiability properties of functions*, 1970.

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14:30–15:15 Computable Π_2 Scott sentences

Julia Knight (University of Notre Dame, USA)

Abstract. Scott [2] showed that for any countable structure $\mathcal{A}$, there is an $L_{\omega_1\omega}$-sentence φ whose countable models are exactly the isomorphic copies of $\mathcal{A}$. We call φ a *Scott sentence* for $\mathcal{A}$.

Montalbán [1] showed that for a countable ordinal $\alpha \geq 1$, $\mathcal{A}$ has a $\Pi_{\alpha+1}$ Scott sentence iff the orbits of all tuples are defined by Σ_α formulas. In particular, for $\alpha = 1$, $\mathcal{A}$ has a Π_2 Scott sentence iff the orbits of all tuples are defined by existential formulas. In $L_{\omega_1\omega}$-formulas, there are countable disjunctions and conjunctions, but only finite strings of quantifiers. *Computable* infinitary formulas are formulas of $L_{\omega_1\omega}$ in which the infinite disjunctions and conjunctions are over c.e. sets. These formulas, while infinite, seem comprehensible.

Some structures, such as the ordered group of rationals and the standard model of arithmetic, have a *computable* Π_2 Scott sentence. Just from the definition, we get necessary and sufficient conditions for this. We give examples illustrating why certain other conditions, in particular, on families of formulas that define the orbits, are either not sufficient, or not necessary. For a special class of structures, in a language with just unary relation symbols, we show that $\mathcal{A}$ has a computable Π_2 Scott sentence iff there is a computable Π_2 formula $\beta(x)$ saying, in all models of the $\forall\exists$-theory of $\mathcal{A}$, that the type of x is isolated.

Joint work with Karen Lange and Charles McCoy CSC.

References

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15:30–15:50 A new way of classifying word problems

Luca San Mauro (Università degli studi di Bari Aldo Moro, Italy)

Abstract. The study of word problems dates back to the work of Dehn in 1911. Given a recursively presented algebra A, the word problem of A is to decide if two words in the generators of A refer to the same element. Much is known about the complexity of word problems for familiar algebraic structures: e.g., the Novikov-Boone theorem, one of the most spectacular applications of computability to general mathematics, states that the word problem for finitely presented groups is unsolvable. Yet, the computability theoretic tools commonly employed to measure the complexity of word problems (e.g., Turing or m-reducibility) are defined for sets, while it is generally acknowledged that many computational facets of word problems emerge only if one interprets them as equivalence relations.

In this work, we revisit the world of word problems, with a special focus on groups and semigroups, through the lens of the theory of equivalence relations, which has grown immensely in recent decades. To do so, we employ computable reducibility, a natural effectivization of Borel reducibility.

This talk collects joint works with Uri Andrews, Valentino Delle Rose, Meng-Che Ho, and Andrea Sorbi.

16:00–16:20 Degrees and applications of e-pointed trees

Josiah Jacobsen-Grocott (University of Wisconsin–Madison, USA)

Abstract. Enumeration pointed trees (e-pointed trees) arise in work by Montalbán [1] in computable structure theory and the study of possible degree spectra of structures. They have since also proved useful in the study of the hyperenumeration degrees.

McCarthy [2] studied e-pointed trees on Cantor space, and showed that the enumeration degrees of e-pointed trees in Cantor space are exactly cototal degrees. He further showed that this characterization remains unchanged if we introduce uniformity or allow for dead ends.

We prove that the enumeration degrees of e-pointed trees in Baire space behave differently. Allowing dead ends gives rise to a very large class of degrees, characterized in terms of hyperenumeration reducibility and containing all Π^1_1 degrees. On the other hand, the class of Baire e-pointed trees without dead ends does not even contain all arithmetic degrees. Nevertheless, it is strictly larger than the class of all cototal degrees. Sandwiched properly in between the cototal degrees and the degrees of e-pointed trees without dead ends are the introenumerable enumeration degrees.

Joint work with Jun Le Goh, Joe Miller, and Mariya Soskova.

References

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17:00–17:20 Generic and coarse computability of Abelian p-groups

Valentina Harizanov (George Washington University, USA)

Abstract. Motivated by asymptotic density problems in combinatorial group theory, and by approximately computable sets introduced and studied by Jockusch and Schupp in computability theory, we formulated the notions of densely computable structures such as generically and coarsely computable structures. We first studied generically and coarsely computable equivalence structures and injection structures, and generically and coarsely computable isomorphisms of these structures. Returning to groups, we now investigate generically computable Abelian p-groups, as well as other variants of approximately computable Abelian p-groups. There is Khisamiev's characterization of reduced Abelian p-groups of certain length with computable isomorphic copies. We define a countable group to be generically computable if it has a computably enumerable subgroup the domain of which is an asymptotically dense set. While every countable Abelian p-group is isomorphic to a generically computable group, this is not the case with Abelian groups in general. We further consider a graded family of elementarity conditions for computably enumerable subgroups that approximate groups. We also investigate coarsely computable Abelian p-groups, which are approximated by computable groups. We show that every countable Abelian p-group is isomorphic to a coarsely computable group.

Joint work with Wesley Calvert and Douglas Cenzer.

References

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[3] W. Calvert, D. Cenzer, V. Harizanov, *Generically computable Abelian groups and isomorphisms*, submitted.

17:30–17:50 On the complexity of some topological properties in highly computable graphs

Valentino Delle Rose (Università di Roma "La Sapienza", Italy)

Abstract. The problem of deciding whether a graph has an *Eulerian* path, namely a path visiting each edge of the graph exactly once, has a very long history in mathematics, dating back to the famous problem of *the seven bridges of Könisberg*, solved by Euler in 1736.

Around 200 years later, Erdős, Grünwald and Vászsonyi extended Euler's result by characterizing those graphs which admit an infinite Eulerian path. This characterization strongly relies on the *number of ends* of a graph, namely, the maximum number of infinite connected components which can be obtained by removing a finite number of edges. Indeed, only graphs with one or two ends can have Eulerian paths.

From the point of view of computability theory, their result highlights an important difference between computable graphs and *highly computable* graphs, the latter notion corresponding to computable and locally finite graphs where, in addition, one can uniformly compute the degree of each vertex. In fact, Bean showed that, while there are computable, locally finite graphs which admit an Eulerian path but no computable one, every highly computable graph admitting Eulerian paths must have a computable one.

However, deciding whether a given graph admits an Eulerian path at all is a difficult problem: as Kuske and Lohrey have shown, such problem is Π^0_2-complete even when restricting to connected, highly computable graphs. Interestingly, this turns out to be the same difficulty of simply counting the number of ends in a highly computable graph.

Motivated by these considerations, we have studied how the difficulties of these two problems precisely relate. We have found that counting the ends of the graph indeed represents the hardest task when deciding the existence of an Eulerian path. More precisely, we have shown that:

(1) deciding existence of Eulerian paths is only (2-c.e.)-complete when restricting to highly computable graphs with one end,
(2) the same problem realizes precisely the m-degrees of Δ^0_2 sets in the case of highly computable graphs with two ends.

To get these results we have conducted a detailed analysis, which we believe of independent interest, of the computational hardness of what we call the *separation problem*: to decide whether a finite set of edges separates a connected and highly computable graph into two or more infinite connected components. Two results here are particularly relevant. On the one hand, we show that any function which takes as input a highly computable graph and outputs a finite set of edges separating the graph must compute the halting problem. On the other hand, the separation problem turns out to be (non-uniformly) decidable for highly computable graphs with finitely many ends: in fact, from the number of ends of a graph and a single maximally separating set, we can compute the whole collection of separating sets for this graph.

Joint work with Nicanor Carrasco-Vargas and Cristóbal Rojas.

References

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[2] P. Erdős, T. Grünwald, E. Vázsonyi, *Über Euler-Linien Unendlicher Graphen*, Journal of Mathematics and Physics, 17 (1938), 59–75.

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18:00–18:20 Revisiting the reverse mathematics of the Tietze extension theorem: preserving suprema

Paul Shafer (University of Leeds, UK)

Abstract. The following version of the Tietze extension theorem is well-known to be provable in RCA_0 [1–2]. Let $\mathcal{C}$ be a closed subset of a complete separable metric space $\mathcal{X}$, and let $f\colon \mathcal{C} \to [-1,1]$ be a continuous function. Then f extends to a continuous $g\colon \mathcal{X} \to [-1,1]$, where $\forall x \in \mathcal{C}\ g(x) = f(x)$. However, in this formulation g need not preserve the supremum of f. That is, it may be that $\sup_{x\in\mathcal{X}} |g(x)| > \sup_{x\in\mathcal{C}} |f(x)|$. We show that the following version of the supremum-preserving Tietze extension theorem requires Π^1_1-CA_0. Let $\mathcal{C}$ be a closed subset of a complete separable metric space $\mathcal{X}$, and let $f\colon \mathcal{C} \to [-1,1]$ be a continuous function. Then f extends to a continuous $g\colon \mathcal{X} \to [-1,1]$, where $\forall x \in \mathcal{C}\ g(x) = f(x)$ and additionally $\forall x \in \mathcal{X}\ \exists y \in \mathcal{C}\ |g(x)| \le |f(y)|$. This is an unusual example of a statement about continuous functions requiring Π^1_1-CA_0.

References

[1] Douglas Kenyon Brown, *Functional Analysis in Weak Subsystems of Second Order Arithmetic*, PhD Thesis, The Pennsylvania State University, 1987.

[2] Stephen G. Simpson, *Subsystems of Second Order Arithmetic*, Second edition, Perspectives in Logic, Cambridge University Press, Cambridge; Association for Symbolic Logic, Poughkeepsie, NY, 2009.

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