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Several Complex Variables: Theory and Applications
Special Session A28

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This special session presents talks on several complex variables (SCV) and its applications to other areas of mathematics and beyond. Complex analysis remains a fundamental tool across various fields–from mathematics to engineering. With recent developments in SCV, new applications have emerged. One such example comes from the study of mapping problems in several complex variables; namely, does there exist a proper, holomorphic mapping from the unit ball in $\mathbb{C}^n$ to the unit ball in $\mathbb{C}^N$ where $n, N \geq 2$? The existence of such monomial mappings between balls is closely related to finding the *sparsest* solution to a certain system of linear equations. Thus, one can study compressed sensing problems in the context of several complex variables. For another application, Wiener-Hopf techniques form an important tool for studying diffraction problems in physics, and extending these techniques in several complex variables is proving to be useful for these applications. For a final example, singular integral operators have been a central object of study in harmonic analysis for many years, and they arise naturally in complex analysis through the Bergman and Szegö projection operators, and the Cauchy-Fantappie singular integral operators. The regularity properties of these operators remain of fundamental interest in the field. The properties of these canonical operators also play crucial roles in operator theory. This special session aims to bring together experts from complex analysis and adjacent fields, fostering cross-pollination of ideas, techniques, and problems.

Schedule and Abstracts

July 23, 2024

11:00–11:20 A lower bound for the essential norm of the Leray transform
David Barrett (University of Michigan, USA)

Abstract. A Berezin-transform-based method for estimating the essential norm of the Leray transform from below will be presented.

11:30–11:50 Waves, oscillatory double integrals, and multidimensional complex analysis
Raphael Assier (University of Manchester, UK)

Abstract. In this talk, I will give an overview of recent developments linking wave theory and multidimensional complex analysis. I will explain how a procedure (developed in [1]) of complex deformation of the integration surface of Fourier-like highly oscillatory double integrals can lead to closed-form far-field asymptotics results in wave diffraction theory. Each far-field component will be shown to be connected to a special point on the singularity set of the integrand. The procedure will be illustrated through the three-dimensional problem of plane wave diffraction by a quarter-plane [2, 3, 4] and the two-dimensional problem of plane wave diffraction by a penetrable wedge [5, 6, 7]. We will also show how it can be used to shed some light on wave propagation in periodic structures [8].

References

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[8] A.V. Shanin, R.C. Assier, A.I. Korolkov, and O.I. Makarov. *A note on double Floquet-Bloch transforms and the far-field asymptotics of Green's functions tailored to periodic structures.* Preprint, arXiv:2402.08076, 2024.

12:00–12:20 On spectrum of Hankel operators on the polydisk
Željko Čučković (University of Toledo, USA)

Abstract. We give sufficient conditions for the essential spectrum of the Hermitian square of a class of Hankel operators on the Bergman space of the polydisk to contain intervals. We also compute the spectrum in case the symbol is a monomial. (Joint work with Zhenghui Huo and Sonmez Sahutoglu).

12:30–12:50 The Commutator of the Bergman Projection
Nathan Wagner (Brown University, USA)

Abstract. Consider a bounded, strongly pseudoconvex domain $D \subset \mathbb{C}^n$ with minimal boundary smoothness (namely, the class C^2) and let b be a locally integrable function on D. We characterize boundedness (resp., compactness) in $L^p(D), p > 1$, of the commutator $[b, P]$ of the Bergman projection P in terms of an appropriate bounded (resp. vanishing) mean oscillation requirement on b. We also establish the equivalence of such notion of BMO (resp., VMO) with other BMO and VMO spaces given in the literature. In particular, we relate dyadic BMO spaces to those defined using Kobayashi metric balls (see [1,4]). Our proofs use a dyadic analog of the Berezin transform and holomorphic integral representations going back (for smooth domains) to N. Kerzman & E. M. Stein, and E. Ligocka in [2,3].

References

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14:30–14:50 Leandro Arosio (University Roma 2 Tor Vergata, ITALY) *Abstract.*

15:00–15:20 Matrix–vector form of Picard–Lefschetz theory
Andrey V. Shanin (M.V.Lomonoso v Moscow State University, RUSSIA)

Abstract. Let be $U(t) = X \setminus \sigma(t)$, where X is a tubular neighborhood of $\mathbb{R}^2$ in the complex domain $\mathbb{C}^2$ of variables $z = (z_1, z_2)$,

$$\sigma(t) = \cup_j \sigma_j(t), \qquad \sigma_j = \{(z_1, z_2) \in X \,:\, g_j(z,t) = 0\},$$

$g_j(z,t)$ are holomorphic functions of variables z and complex parameters $t = (t_1, t_2, \dots)$. We assume that $g_j(z,t)$ take real values when z and t are all real. Let $\hat{U}(t)$ be a universal covering of $U(t)$. We study the ramification of $H_2(\hat{U}(t))$ as t goes along some closed loop λ bypassing about the Landau set of the singularities. It is well-known that this ramification is described by the Picard–Lefschetz theory [1], however, practical application of this theory may be not simple.

A matrix–vector form of Picard–Lefschetz theory is proposed for this case. A relative homology group

$$\hat{H}_2(X, \sigma(t)) = \psi^{-1}(H_2(X, \sigma(t))),$$

is introduced, where $\psi : \hat{U} \to U$ is a canonical projection. The Picard–Lefschetz results are provided by two theorems presented in the talk. The first theorem expresses the ramification of the elements of $\hat{H}_2(X, \sigma(t))$ in a simple matrix-vector form. The second theorem shows that the elements of $H_2(\hat{U}(t))$ can be obtained from a certain subset of the elements of $\hat{H}_2(X, \sigma(t))$ using an "inflation" procedure. Thus, ramification of $H_2(\hat{U}(t))$ becomes expressed in the matrix–vector form.

The method proposed in the talk is aimed for development into a 2D Wiener–Hopf method (which is an ambitious task indeed). The elements of $H_2(\hat{U}(t))$ are possible integration surfaces Γ for 2D Sokhotsky formulae of the form

$$f_s(t_1, t_2) = \frac{1}{(2\pi i)^2} \int_\Gamma \frac{f(z_1, z_2)}{(z_1 - t_1)(z_2 - t_1)} dz_1 \wedge dz_2,$$

$\sigma_j(t)$ are the lines $z_1 = t_1$, $z_2 = t_2$, and the singularities of $f(z)$. A ramification of Γ corresponds to ramification of the integral and thus of $f_s(z)$, which is of a great importance for applications.

15:30–15:50 On extrapolation of analytic functions
Dmitry Ponomarev (Centre Inria d'Université Côte d'Azur, FRANCE)
Abstract. The issue of extrapolation of an analytic function is known to be an ill-posed problem and as such it involves a regularisation strategy. We consider some instances of the problem of analytic continuation problem in a basic one-complex-variable setting. We discuss an efficient procedure of reconstruction of an H^2 function from its partial (and potentially non-exact) boundary values and connections to Carleman's formulas. The latter can yield a generalisation to multidimensional settings. If time permits, I will also show some practical situations of extrapolation in certain subspaces of harmonic functions from partial knowledge of a function inside its domain.

16:00–16:20 Spherical Sommerfeld Integrals and Monodromy
Valentin Kunz, (Università di Bologna, ITALY)
Abstract. In this talk we give an overview on some recent work involving the application of several complex variables to the 'quarter-plane problem'. That is, the physical problem resulting from the interaction of a time-harmonic acoustic wave-field with a 'sound-hard' or 'sound-soft' quarter of a plane in three spatial dimensions; see [1].

After formulating the corresponding boundary value problem (BVP), which models this interaction, we reduce it to an Eigenvalue problem of the Laplace-Beltrami operator on a ramified covering of the 2-sphere $\mathbb{S}^2 = \{\boldsymbol{x} \in \mathbb{R}^3 |\ x_1^2 + x_2^2 + x_3^2 = 1\}$. This reformulation of the BVP enables us to utilise the theory of several complex variables: We complexify the covering space's base-space, the 2-sphere, and study the propagation of singularities (which include the real ramification points) within the complexification $\mathbb{S}^2_{\mathbb{C}} = \{\boldsymbol{z} \in \mathbb{C}^3 |\ z_1^2 + z_2^2 + z_3^2 = 1\}$. This approach was pioneered in the euclidean setting by Assier and Shanin in [2]. Here, we employ a *local* analytical continuation procedure which allows us to represent the *global* physical wave-field (i.e., the non-complexified wave-field) as a contour integral on some 'sphere at infinity' (this notion is made precise by viewing $\mathbb{S}^2_{\mathbb{C}}$ as a projective variety). This generalises the concept of Sommerfeld

integrals from the (complexified) circle (which is directly linked to the classical, one-complex-variable, Wiener-Hopf technique) to the (complexified) 2-sphere. Our 'spherical Sommerfeld integral' involves some unknown spectral functions, which are defined on the sphere at infinity and involve only a *single* complex variable. Conversely, these spectral functions can be expressed as a contour integral on the original 2-sphere $\mathbb{S}^2$ which serves as the base-space of our covering. This allows us to derive a monodromy representation which describes the singularity structure of our spectral functions. In turn, this allows us to link the spectral functions to the theory of Fuchsian ODE's (this is Hilbert's 21^{st} problem). Time permitting, we will discuss further aspects of our theory and potential future directions.

17:00–17:20 Quasi-finite typeness and 1-regular types on algebraic CR manifolds: global boundedness I
Bernhard Lamel (Texas A&M University-Qatar, QATAR)
Abstract. We first review the well-known notion of regular types on real hypersurfaces or CR manifolds, i.e. the order of contact of non-singular holomorphic curves with such manifolds. We discuss the global boundedness properties of such a type for a manifold and show that such a type is always uniformly bounded on any (not necessarily bounded) real-algebraic manifold containing no complex-analytic curves. Such a result follows from introducing and studying a more general notion of quasi-finite typeness for real-analytic maps and establishing its global boundedness for arbitrary real-algebraic/Nash maps. This is joint work by B. Lamel, N. Mir and G. Rond.

References

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17:30–17:50 Quasi-finite typeness and 1-regular types on algebraic CR manifolds: global boundedness II
Nordine Mir (Texas A&M University-Qatar, QATAR)
Abstract. We first review the well-known notion of regular types on real hypersurfaces or CR manifolds, i.e. the order of contact of non-singular holomorphic curves with such manifolds. We discuss the global boundedness properties of such a type for a manifold and show that such a type is always uniformly bounded on any (not necessarily bounded) real-algebraic manifold containing no complex-analytic curves. Such a result follows from introducing and studying a more general notion of quasi-finite typeness for real-analytic maps and establishing its global boundedness for arbitrary real-algebraic/Nash maps. This is joint work by B. Lamel, N. Mir and G. Rond.

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July 24, 2024

11:30–11:50 Bergman logarithmically flat and obstruction flat CR manifolds
Peter Ebenfelt (University of California San Diego, USA)
Abstract. Let $\Omega \subset \mathbb{C}^n$ be a smoothly bounded, strictly pseudoconvex domain. The boundary $\partial\Omega$ is said to be Bergman logarithmically flat if the log singularity in Fefferman's asymptotic expansion of the Bergman kernel vanishes (to infinite order). It is called obstruction flat if the log singularity (the obstruction function) of the Cheng-Yau log-potential of the complete Kähler-Einstein metric in Ω vanishes. The Ramadanov Conjecture asserts that if $\partial\Omega$ is Bergman logarithmically flat, then it is spherical. There is a similar conjecture for obstruction flat boundaries. Both conjectures, suitably reformulated, fail for domains in more general complex manifolds in higher dimension $n \geq 3$, but the situation is still unclear for domains in $\mathbb{C}^n$ (for $n \geq 3$). In this talk, we shall present recent work and open questions concerning these conjectures and the general structure of Bergman logarithmically flat and obstruction flat CR manifolds.

12:00–12:20 The $\overline{\partial}$-problem in $Z(q)$-domains
Debraj Chakrabarti (Central Michigan University, USA)
Abstract. A smoothly bounded, relatively compact domain Ω in a complex manifold M of dimension n is said to satisfy condition $Z(q)$ for some $1 \leq q \leq n-1$ if the Levi form of the boundary $\partial\Omega$ has at least $n-q$ positive or at least $q+1$ negative eigenvalues. It is well-known that this condition implies the Sobolev-$\frac{1}{2}$ estimates on the $\overline{\partial}$-problem and consequently the L^2-cohomology $H^{p,q}_{L^2}(\Omega)$ is finite dimensional with respect to any Hermitian metric on M. We consider the problem of giving sufficient biholomorphically invariant geometric conditions for the vanishing of this cohomology. Such conditions involve the interaction of the partial convexity of the boundary (given by the Levi form) and the interior convexity of the domain Ω (given by the complex Hessian of a weight function), where the interaction is encoded in a smoothly varying bundle of common positive directions at the boundary. We also discuss the sufficiency of these conditions.

12:30–12:50 The Julia-Wolff-Carathéodory Theorem in convex domains of finite type
Matteo Fiacchi (University of Ljubljana, SLOVENIA)
Abstract. The classical Julia-Wolff-Carathéodory Theorem says that, if f is a holomorphic self-map of the unit disk $\mathbb{D}$ and $\xi \in \partial\mathbb{D}$ such that

$$\liminf_{z\to\xi} \frac{1-|f(z)|}{1-|z|} < +\infty$$

then the derivative f' has non-tangential limit at ξ and the limit value can be computed in terms of the Poincaré distance. The Theorem has been generalized to several complex variables by Rudin in the unit ball, by Abate in strongly convex and by Abate-Tauraso in convex domains of finite type, under some technical assumptions. In this talk we will present a version of the Julia-Wolff-Carathéodory Theorem for a holomorphic map $f : D \to D'$ between convex domains of finite type. In particular, given a point $\xi \in \partial D$ with finite dilation we show that the K-limit of f at ξ exists and is a point $\eta \in \partial D'$, and we obtain asymptotic estimates for all entries of the Jacobian matrix of the differential df_z in terms of the multitypes at ξ and at η. Moreover, we introduce a generalization of Bracci-Patrizio-Trapani's pluricomplex Poisson kernel which, together with the dilation at ξ, gives a formula for the restricted K-limit of the normal component of the normal derivative $\langle df_z(n_\xi), n_\eta\rangle$. This is a joint work with L. Arosio.

14:30–14:50 CR functions at CR singularities: approximation, extension, and hulls
Jiri Lebl (Oklahoma State University, USA)
Abstract. We study three possible definitions of the notion of CR functions at CR singular points, their extension to a fixed-neighborhood of the singular point, and analogues of the Baouendi–Trèves approximation in a fixed neighborhood. In particular, given the existence of a large enough disc hull shrinking to a point, we find the fixed-neighborhood extension and hence approximation properties. We provide examples showing the distinctions between the classes and the various properties studied.

15:00–15:20 Kähler-Einstein Bergman metrics on pseudoconvex domains
Ming Xiao (University of California San Diego, USA)
Abstract. A well-known conjecture of Yau asserts that the Bergman metric on a bounded pseudo-convex domain in $\mathbb{C}^n$ is Kähler-Einstein if and only if the domain is homogeneous. A special case of this conjecture was posted earlier by Cheng: if the Bergman metric of a smoothly bounded strongly pseudoconvex domain is Kähler-Einstein, then the domain is biholomorphic to the unit ball. In this talk, we will discuss old and new results concerning the conjectures of Cheng and Yau.

15:30–15:50 Potential theory and exceptional sets for the Drury Arveson space
Nikolaos Chalmoukis (University Milano-Bicocca , ITALY)
Abstract. In this talk we will introduce an energy functional for measures on the boundary of the unit ball of $\mathbb{C}^n$ which is appropriate for the study of function theory in the Drury Arveson space. In particular , the energy functional leads to a notion of capacity for sets on the boundary

of the unit ball. Using this notion of capacity we characterize radial or admissible exceptional sets for the Drury Arveson space, to- tally null sets and we give a stronger necessary condition for a function to be cyclic. Hence, in some sense, the proposed notion of capacity plays the same role that linear Lebesgue measure plays in the theory of the Hardy space in the unit disc, or logarithmic capacity plays for the classical Dirichlet space. The talk is based on a joint work with M. Hartz.

16:00–16:20 Bergman metrics of constant holomorphic sectional curvature
John Treuer (University of California San Diego, USA)
Abstract. In 2023, Huang and Li considered complex manifolds admitting a Bergman metric of constant holomorphic sectional curvature. Building on their work, in this talk we show no complex manifold whose Bergman space is base-point free, separates directions and separates points can have a Bergman metric with identically zero holomorphic sectional curvature.

17:00–17:20 On the Hunag-Ji-Yau algebraicity Conjecture
Ilya Kossovkiy (Masaryk University, CZECHIA)
Abstract. In our joint work with Jan Gregorovic, we investigate the problem of holomorphic algebraizibility for real hypersurfaces in complex space. We introduce a new invariant of a (real-analytic) Levi-nondegenerate hypersurface called *the jet transcendence degree*. Using this invariant, we solve in the negative the Conjecture of Huang, Ji and Yau on the algabraizability of real hypersurfaces with algebraic syzygies.

17:30–17:50 On (some classes of) Levi-nondegenerate homogeneous CR manifolds
Costantino Medori (University of Parma, ITALY)
Abstract. The notion of Levi-nondegenerate of order k for CR manifolds is based on the higher order Levi form introduced by M. Freeman in [2], generalising the classical notion of Levi-nondegenerate CR manifold (which corresponds to $k = 1$).

For homogeneous CR manifolds we analysed Levi-nondegeneracy in terms of CR-algebras introduced in [3] (see also [4]). Generic Levi-nondegenerate CR manifolds have finite dimensional algebras of automorphisms (see, for instance, [5]).

In this talk we will consider the submanifolds M of a complex flag manifold $F = G/P$ of a semisimple Lie groups G that are orbits of a real forms $G_{\mathbb{R}}$ (see [1]). We analyse Levi-nondegenerate submanifolds, showing that they have order $k \leq 3$ (see [6]). In particular, the compact orbits have order at most 2 (see [7]). These estimates are sharp.

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18:00–18:20 Hardy Space and Szegő projection on quotient domains
Liwei Chen (Syracuse University, USA)

Abstract. In this talk, we will briefly introduce the quotient domain setup in the theory of Hardy space and Szegő projection. Then we will study the analytic behavior of the Szegő projection on some non-smooth domains as applications.

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