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Harmonic Analysis and Geometric Measure Theory
Special Session A11

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Abstract. This session aims at exploring the recent progress in and interplay between a few of the most prominent families of problems in Harmonic Analysis, such as

- Fourier restriction inequalities, decoupling and local smoothing estimates;
- problems related to the set and maximal Kakeya conjecture;
- singular integrals in the Euclidean, doubling, and non-doubling setting, and their application to Geometric Measure theory;
- Continuous and discrete Radon transforms, oscillatory integrals related to Carleson's theorem, and discrete operators in Harmonic Analysis.

These circles of problems, and the researchers working on them, all rely on diverse technical tools from a variety of fields of contemporary mathematics such as modern Analysis, Algebraic Geometry, Combinatorics, and Number Theory. In contrast with the increasingly specialized nature of their problems, researchers coming from different subfields share the same language of classical Harmonic Analysis and Geometric Measure Theory. This special session aims at highlighting both shared aspects, and at fostering interaction between researchers from each subfield. The invited speakers draw from both pools of world-leaders in the field and of innovative, accomplished and diverse earlier-career mathematicians. The Italian and American Harmonic Analysis communities will both be well represented, with additional emphasis on geographic diversity.

Schedule and Abstracts

July 23, 2024

11:00–11:20 Singular integrals along variable codimension one subspaces
Luz Roncal (Basque Center for Applied Mathematics - BCAM, SPAIN)
Abstract. The talk concerns maximal operators on $\mathbb{R}^n$ defined by taking arbitrary rotations of tensor products of an $(n-1)$-dimensional Hörmander–Mihlin multiplier with the identity in one coordinate. These maximal operators are connected to differentiation problems and maximally modulated singular integrals such as Sjölin's generalization of Carleson's maximal operator.

As the main result, we prove a weak-type $L^2(\mathbb{R}^n)$ estimate on band-limited functions. As corollaries, we obtain a sharp $L^2(\mathbb{R}^n)$ estimate for the maximal operator restricted to a finite set of rotations in terms of the cardinality of the finite set and a version of the Carleson–Sjölin theorem. Our approach relies on higher dimensional time-frequency analysis elaborated according to the directional nature of the operator under study.

This is joint work with Odysseas Bakas, Francesco Di Plinio, and Ioannis Parissis.

11:30–11:50 Localised variants of multilinear restriction
David Beltran (Universitat de València, SPAIN)
Abstract. The multilinear Fourier Restriction estimates for hypersurfaces of Bennett–Carbery–Tao have played a central role in many recent developments in Euclidean Harmonic Analysis. Recently, Bejenaru introduced some variants in which a gain in the estimates is possible if the functions are Fourier localised to a neighbourhood of a submanifold of the hypersurfaces involved. In this talk, we present an alternative approach to that of Bejenaru for obtaining such

refined estimates which, in particular, connects them to the theory of Kakeya–Brascamp–Lieb inequalities. Furthermore, we will present generalisations of Bejenaru's estimates under finer localisation conditions on the functions. This is joint work with Jonathan Bennett and Jennifer Duncan.

12:00–12:20 Analysis on trees with nondoubling flow measures
Maria Vallarino (Politecnico di Torino, ITALY)

Abstract. We consider infinite nonhomogeneous trees endowed with nondoubling flow measures of exponential growth. In this setting we develop a Calderón–Zygmund theory, we construct a family of dyadic sets, and we define BMO and Hardy spaces, proving a number of desired results extending the corresponding theory as known in the classical setting. This construction was inspired by a paper by Hebisch and Steger on homogeneous trees equipped with the canonical flow.

Moreover, given a nonhomogeneous tree T equipped with a flow measure m and a flow Laplacian $\mathcal{L}$ which is self-adjoint on $L^2(m)$, we prove weighted L^1-estimates for the heat kernel of $\mathcal{L}$ and its gradient. Such estimates combined with the Calderón–Zygmund theory adapted to the space (T, m) can be applied to study the boundedness properties of the Riesz transform associated with $\mathcal{L}$ and of spectral multipliers of $\mathcal{L}$.

This is a joint work with Matteo Levi, Alessio Martini, Federico Santagati and Anita Tabacco.

12:30–12:50 Multi-parameter potential tehory: results and problems.
Nicola Arcozzi (University of Bologna, ITALY)

Abstract. Multi-parameter potential theory might be seen as the tensor-product version of classical potential theory. It naturally arises, for instance, in the study of function spaces on poly-discs. Despite this short description, progress in the area has been slow and many basic questions remain unanswered. I will present a (biased) survey of old and recent results, and some problems which are central to have further progress. Some of the recent results were obtained by I. Holmes, P. Mozolyako, K.M. Perfekt, G. Sarfatti, A. Volberg, and myself.

14:30–14:50 Oscillatory integrals over locally compact fields: A unified theory- part I
Jim Wright (School of Mathematics, University of Edinburgh, UK)

Abstract. Here we consider oscillatory integrals defined over general locally compact fields $\mathbb{K}$. When $\mathbb{K} = \mathbb{R}$ is the real field, oscillatory integrals are a basic object of study in harmonic analysis. On the other hand, complete exponential sums or character sums can be realised as oscillatory integrals over the p-adic field $\mathbb{K} = \mathbb{Q}_p$. These are basic objects in number theory. In both cases, for real oscillatory integrals and exponential sums, there is an extensive literature giving sharp bounds for these oscillating entities.

In this talk, we discuss a unified theory for oscillatory integrals defined over any locally compact field. This is joint work with Gian Maria Dall'Ara.

15:00–15:20 Oscillatory integrals over locally compact fields: A unified theory- part II
Gian Maria Dall'Ara (Indam & Scuola Normale Superiore, ITALY)

Abstract. Here we consider oscillatory integrals defined over general locally compact fields $\mathbb{K}$. When $\mathbb{K} = \mathbb{R}$ is the real field, oscillatory integrals are a basic object of study in harmonic analysis. On the other hand, complete exponential sums or character sums can be realised as oscillatory integrals over the p-adic field $\mathbb{K} = \mathbb{Q}_p$. These are basic objects in number theory. In both cases, for real oscillatory integrals and exponential sums, there is an extensive literature giving sharp bounds for these oscillating entities.

In this talk, we discuss a unified theory for oscillatory integrals defined over any locally compact field. This is joint work with Jim Wright.

15:30–15:50 Quantitative norm convergence of triple ergodic averages for commuting transformations
Polona Durcik (Chapman University, USA)

Abstract. We discuss a quantitative result on norm convergence of triple ergodic averages with respect to three general commuting transformations. For these averages we prove an r-variation estimate, $r > 4$, in the norm. We approach the problem via real harmonic analysis, using the recently developed techniques for bounding singular Brascamp-Lieb forms. It remains an open problem whether the analogous norm-variation estimates hold for all $r \geq 2$ as in the cases of one or two commuting transformations, or whether such estimates hold for any $r < \infty$ for more than three commuting transformations. This is joint work with Lenka Slavíková and Christoph Thiele.

16:00–16:20 The oscillatory bi-est operator
Cristina Benea (Nantes Université, FRANCE)

Abstract. The oscillatory bi-est operator is yet another example of a modulation-invariant multilinear operator which presents certain curvature features, since it consists of the classical bi-est operator and carries an oscillatory factor in the form of a complex exponential. This is a joint work with I. Oliveira, and builds up on related works with F.Bernicot, V. Lie, M. Vitturi.

17:00–17:20 On the curved trilinear Hilbert transform
Bingyang Hu (Auburn, USA)

Abstract. The goal of this talk is to discuss the L^p boundedness of the trilinear Hilbert transform along the moment curve. The main difficulty in approaching this problem(compared to the classical approach to the bilinear Hilbert transform) is the lack of absolute summability after we apply the time-frequency discretization (which is known as the LGC methodology introduced by V. Lie in 2019).

To overcome such a difficulty, we develop a new, versatile approach – referred to as *Rank-II LGC* (which is also motivated by the study of the non-resonant bilinear Hilbert-Carleson operator by C. Benea, F. Bernicot, V. Lie, and M. Vitturi in 2022), whose control is achieved via the following interdependent elements:

1). a sparse-uniform decomposition of the input functions adapted to an appropriate time-frequency foliation of the phase-space;
2). a structural analysis of suitable maximal "joint Fourier coefficients";
3). A level set analysis with respect to the time-frequency correlation set.

This is joint work with Victor Lie (Purdue).

17:30–17:50 Fitting Smooth Functions to Data
Charles Fefferman (Princeton, USA)

Abstract. The talk will discuss the following problems, which my friends and I have studied over the last 20 years or so:

Let E be an arbitrary subset of $\mathbb{R}^n$. How can we decide whether a given function $f : E \to \mathbb{R}$ extends to a C^m function on $\mathbb{R}^n$ for fixed m? If such an extension F exists, then how small can we take its C^m norm? What can we say about F and its derivatives at points in or near E? Can we take F to depend linearly on f for fixed E and m?

Suppose E is finite. Can we compute an F with C^m norm of smallest possible order of magnitude? How many computer operations does it take? What if we demand merely that F agree with f on E to a given tolerance, rather than demanding that $F = f$ on E? What if we are allowed to delete a few points of E as "outliers"? Which points should we delete?

What if F is required to satisfy constraints, e.g. $F \geq 0$ or F convex? What if the C^m norm is replaced by a Sobolev norm?

July 24, 2024

11:30–11:50 Hardy spaces and dilations on homogeneous groups
Tommaso Bruno (University of Genova, ITALY)

Abstract. On a homogeneous Lie group, I will discuss a characterization of the one-parameter groups of dilations whose associated Hardy spaces in the sense of Folland and Stein are the same. The talk will be based on a joint work with J. T. van Velthoven [BvV].

12:00–12:20 Uniform bounds for trilinear Fourier multiplier forms
Olli Saari (Universitat Politecnica de Catalunya, SPAIN)

Abstract. Let $d \geq 1$. Given three test functions f_1, f_2 and f_3 on $\mathbb{R}^d$ and a symbol function $m : \mathbb{R}^{3d} \to \mathbb{C}$, consider the trilinear form

$$\Lambda(f_1, f_2, f_3) = \iiint_{\xi_1+\xi_2+\xi_3=0} \widehat{f}_1(\xi_1)\widehat{f}_2(\xi_2)\widehat{f}_3(\xi_3) m(\xi_1, \xi_2, \xi_3)\, d\xi_1 d\xi_2 d\xi_3.$$

Assume the symbol function to be smooth except for a singularity at a d-dimensional linear subspace $\Gamma \subset \mathbb{R}^{3d}$, which is the image of $\{(\xi, \xi, \xi) : \xi \in \mathbb{R}^d\}$ under a map $L = L_1 \oplus L_2 \oplus L_3$ whose blocks L_i satisfy $K^{-1}|x|^2 \leq L_i x \cdot x \leq K|x|^2$ for some K and all $i \in \{1, 2, 3\}$ and all $x \in \mathbb{R}^d$. Under the natural derivative bounds on m (relative to the above mentioned singularity), we show that the form is bounded on $L^{p_1}(\mathbb{R}^d) \times L^{p_2}(\mathbb{R}^d) \times L^{p_3}(\mathbb{R}^d)$ provided that the exponents are in the local $L^2(\mathbb{R}^d)$-range and satisfy the natural scaling conditions

$$\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1, \quad 2 < p_1, p_2, p_3 < \infty.$$

The bound on the form depends on the matrices L_1, L_2 and L_3 only through the parameter K (which is invariant under scalings). In particular, the result recovers the uniform bounds in local L^2-range for the bilinear Hilbert transform and extends them to the two dimensional bilinear singular integrals such as the bilinear variant of the Beurling transform. This is based on joint work with Marco Fraccaroli and Christoph Thiele.

12:30–12:50 Endpoint estimates for Fourier multipliers with Zygmund Singularities
Ioannis Parissis (University of Basque Country & Ikerbasque, SPAIN)

Abstract. We consider Fourier multipliers on the real line with frequency singularities along thin sets: the prototypical examples are given by Hörmander-Mihlin multipliers which are singular along higher-order lacunary sets and corresponding square functions, and by Marcinkiewicz-type multipliers. For these operators we prove the sharp endpoint mapping properties, generalizing previous results of Tao and Wright. Underlying the proofs of such results is a family of generalized Zygmund-type inequalities which are related to the (dual version of the) Chang-Wilson-Wolff inequality. A further and more sophisticated approach will allow us to develop the theory for general sets of singularities that satisfy suitable Zygmund-type assumptions, leading to the full range of (weighted) L^p-estimates, sparse bounds, and a characterization of the Littlewood-Paley property.

14:30–14:50 On the non-resonant Carleson–Radon transform
Martin Hsu (Purdue University, USA)

Abstract. In this talk, we discuss the L^p boundedness of the 2D non-resonant Carleson–Radon transform:

$$CR(f)(x, y) := \text{p. v.} \int_{\mathbb{R}} f(x - t, y - t^2)\, \frac{e^{i\, a(x,y)\, t^3}}{t}\, dt\,, \quad (x, y) \in \mathbb{R}^2\,, \tag{1}$$

where $a(x, y)$ is an arbitrary real measurable function. We prove the following:

Theorem 1. *For $p \in (1, \infty)$, there's a constant C_p independent of $a(x, y)$ such that*

$$\|CR(f)\|_{L^p} \leq C_p \|f\|_{L^p}\,.$$

Previously known results addressed only the restricted situation with $a(x, y) = a(x)$ or $a(x, y) = a(y)$. Our theorem is the first instance treating (1) in full generality.

Our approach relies on the LGC method and involves the following key elements:

- a sparse-uniform dichotomy of the input function adapted to appropriate time-frequency foliation of the phase-space;

- a joint structural analysis of the linearizing stopping-time function $a(x, y)$ in relation to the Gabor coefficients of the input;
- a level set analysis on the time-frequency correlation set.

This is joint work with Victor Lie.

15:00–15:20 On integer distance sets
Marina Iliopoulou (University of Athens, GREECE)

Abstract. An integer distance set is a set in the Euclidean plane with the property that all pairwise distances between its points are integers. In this talk we will explain that any integer distance set lies on a single line or circle, apart from perhaps a small number of its points. This helps us address some questions by Erdős on the size of integer distance sets. For instance, we deduce that integer distance sets in general position are very sparse. The main idea of the proof is transforming the points in an integer distance set into lattice points on some high-degree variety, and then using existing results from number theory to control the number of these lattice points. This is joint work with Rachel Greenfeld and Sarah Peluse.

15:30–15:50 Discrepancy on the sphere with respect to caps of fixed radius
Dmitriy Bilyk (University of Minnesota, USA)

Abstract. A classical result of J. Beck guarantees that the optimal discrepancy of an N-point set in $\mathbb{S}^d$ over all spherical caps with unrestricted radii is of the order $N^{-\frac{1}{2}-\frac{1}{2d}}$. However, it was completely unclear what happens if one fixes the radius. We provide a partial answer to this question by describing a set of radii for which the above bound continues to hold for spherical caps of a fixed radius. To this end we introduce gegenbadly approximable numbers (an analog of badly approximable numbers in diophantine approximations) for which the values of Gegenbauer polynomials stay away from zero in a certain quantitative sense. We also discuss other versions of this question in various settings (e.g. balls in the unit cube or torus), an interesting role played by the case $d \equiv 1 \mod 4$, as well as the so-called 'freak theorem' about continuous functions which have mean zero over all spherical caps of a given radius. The talk is based on joint work with M. Mastrianni and S. Steinerberger.

16:00–16:20 Dimension-free Remez Inequalities, norm designs, and learning big matrices
Alexander Volberg (MSU and Hausdorff Center, USA/GERMANY)

Abstract. Suppose you wish to find a $2^n \times 2^n$ matrix by asking this matrix question that it honestly answers. For example you can ask question "What is your $(1, 1)$ element?" Obviously you will need exponentially many questions like that. But if one knows some information on Fourier side one can ask only log n questions if they are carefully randomly chosen. Of course one pays the price: first of all one would find the matrix only with high confidence (high probability bigger than $1 - \delta$), secondly the error ϵ. I will explain how this can be done using harmonic analysis and probability. The main ingredient is dimension free Remez inequality.

The classical Remez inequality bounds the supremum of a bounded-degree polynomial on an interval X by its supremum on any subset $Y \subset X$ of positive Lebesgue measure. There are many multivariate generalizations of the Remez inequality, but they have constants that depend strongly on dimension. Here we show that a broad class of domains X and test sets Y—termed *norm designs*—enjoy dimension-free Remez-type estimates.

Theorem 2. *Let $n \geq 1$, $\eta > 0$ and $K \geq 2$. Consider $Y = \prod_{j=1}^n Z_j$ for sets $Z_1, Z_2, \ldots, Z_n \subset \mathbf{D}$ such that for all $1 \leq j \leq n$ we have $|Z_j| = K$ and*

$$\min_{z \neq z' \in Z_j} |z - z'| \geq \eta\,. \tag{2}$$

Then for any analytic polynomial $f : \mathbf{D}^n \to \mathbb{C}$ of degree d and individual degree $K - 1$,

$$\|f\|_{\mathbf{D}^n} \leq C(d, K)\|f\|_Y. \tag{3}$$

Here constant $C(d, K) = C(d, K, \eta) = C(K, \eta)^d$, and $C(K, \eta) > 0$ depends only on K and η, and not on dimension n. Moreover if all $Z_j = \Omega_K := \{e^{2\pi i k/K} : k = 0, 1, \ldots, K - 1\}$ then $C(d, K) \leq \big(\mathcal{O}(\log K)\big)^{2d}$.

This is the joint work with Lars Becker, Ohad Klein, Joseph Slote, and Haonan Zhang.

17:00–17:20 On Implicitly Oscillatory Multilinear Integrals
Michael Christ (Berkeley, USA)
Abstract. For $1 \leq j \leq N$ let $\varphi_j : B \to \mathbb{R}^1$ be C^1 surjections, and let $B \subset \mathbb{R}^2$ be an open ball. Consider scalar-valued multilinear forms

$$T(f_1, \ldots, f_N) = \int_{\mathbb{R}^2} \prod_{j=1}^{N} (f_j \circ \varphi_j)\, \eta$$

where η is a smooth compactly supported cutoff function. We aim to prove *a priori* multilinear smoothing inequalities of the form

$$|T(f_1, \ldots, f_N)| \leq C \prod_j \|f_j\|_{W^{p,s}}$$

where $p < \infty$, $W^{p,s}$ is the Sobolev space of functions with s derivatives in L^p, and crucially, s is strictly negative. The emphasis is on the existence of p, s, not on optimal thresholds.

Such an inequality broadly asserts that if at least one factor has Fourier transform supported at high frequencies, then $|T(f_1, \ldots, f_N)|$ is small — hence the term "implicitly oscillatory".

The first result of this type was due to Bourgain for a particular example with $N = 3$. A compactness theorem, without any quantitative inequality, was proved by Joly, Métivier, and Rauch, again for $N = 3$.

We state two results under a mild transversality hypothesis on the mappings φ_j. Firstly, for $N = 4$ and C^ω mappings φ_j, the inequality holds if and only if there does not exist a resonance – that is, a nonconstant C^ω solution $(g_1, \ldots, g_4)$ of $\sum_j (g_j \circ \varphi_j) \equiv 0$ in some connected open subset of $\mathbb{R}^2$.

Secondly, for $N = 3$, the inequality holds for $C^{1,\alpha}$ mappings for any $\alpha > 0$, under a certain nonlinearizability hypothesis on the 3-web in $\mathbb{R}^2$ defined by $(\varphi_j : 1 \leq j \leq 3)$. This hypothesis is a version of nonvanishing curvature of the web, but requires only C^1 regularity, whereas the classical definition of 3-web curvature involves third order derivatives.

Certain sublevel set inequalities are a key element for the proofs of both results. Other elements include local Fourier expansion aka microlocal analysis, stationary phase, and decomposition into structured and pseudorandom summands in the spirit of much work in additive combinatorics. Due to the time constraint, little can be said about the proofs in this talk.

17:30–17:50 A numerical method for the solution of boundary value problems on convex planar domains
Loredana Lanzani (University of Bologna, ITALY)
Abstract. The Unified Transform Method (UTM) was pioneered in the early 1990s by A. S. Fokas and I. M. Gèlfand in their study of the numerical solution of boundary value problems for elliptic PDEs and for a large class of nonlinear PDEs. The UTM provides a connection between the Fourier Transform method for linear PDEs (FT) and its nonlinear counterpart, namely the Inverse Spectral method - also known as Non Linear Fourier Transform method (NLFT).

At the heart of the matter is a new derivation of the FT method for linear equations in one and two (space) variables that follows the same conceptual steps needed to implement the NLFT method for a class of nonlinear evolution equations, thus pointing to a unified approach to the numerical solution of linear and nonlinear PDEs.

From the very beginning, the UTM has attracted a great deal of interest in the applied mathematics community. A multitude of versions of the original method have since been developed, each dealing with a specific family of equations. Here we focus on a 2003 result of A.S. Fokas and A.A. Kapaev pertaining to the study of boundary value problems for the Laplacian on convex polygons: their original approach relied on a variety of tools (spectral analysis of a parameter-dependent ODE; Riemann-Hilbert techniques, etc.) but it was later observed by D. Crowdy that the method can be recast within a complex function-theoretic framework which, in turn,

expands the applicability to so-called circular domains (domains bounded by arcs of circles, with line segments being a special case).

We extend the original approach of Fokas and Kapaev for polygons, to arbitrary convex domains. It turns out that ellipses (which are not circular in the sense of Crowdy) are of particular relevance in applications to engineering because the most popular heat exchangers (namely the shell-and-tube exchangers) have elliptical cross section.

This is joint work with J. Hulse (Syracuse University), S. Llewellyn Smith (UCSD & Scripps Institute of Oceanography) and Elena Luca (The Cyprus Institute).

18:00–18:20 TBA
Jill Pipher (Brown, USA)

Abstract. TBA

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