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BOOK OF ABSTRACTS

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Derivative-Enhanced Pseudo-Labeling for Predictor-Corrector PINNs

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Physics-Informed Neural Networks (PINNs) have emerged as an effective framework for solving partial differential equations by incorporating the underlying physical laws directly into the learning process. However, their performance may be affected by training instability, uncertain convergence, and a strong reliance on the availability and quality of observational data. To address these challenges, we propose a Predictor-Corrector PINN framework that progressively enhances the solution through a sequence of training stages. At each stage, the framework exploits the current prediction to perform physically consistent data augmentation, generating additional informative samples that are subsequently used to refine the solution. This iterative process allows the prediction to be progressively improved while preserving consistency with the governing physical model. The numerical results demonstrate that the proposed approach can provide more stable and accurate solutions, highlighting its potential to improve the robustness and reliability of PINNs in scenarios where domain information is indirect or only partially observed.

Neural Preconditioning Strategies for Mixed-Dimensional PDEs: from Static to Dynamic Training

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Mixed-dimensional partial differential equations (PDEs), characterized by couplings between operators defined on domains of different dimensionality, pose significant computational challenges due to their inherent ill-conditioning. Moreover, the computational cost increases substantially when such problems must be solved repeatedly to address multiple instances of the low-dimensional component, which in many practical applications correspond to varying configurations of fracture, fiber, or vascular networks.

In this work, we develop a novel training strategy for neural preconditioners developed in [1] that leverages the geometric structure of Krylov subspace methods. The core contribution is the formulation of a dynamic training algorithm in which the loss functional directly optimizes the angles between Krylov subspaces that govern the convergence behavior of iterative solvers [2].

The proposed approach retains the advantages of unsupervised learning, such as straightforward data generation and independence from ground-truth solutions, while introducing a transparent, solver-integrated performance metric. A differentiable formulation of the Flexible GMRES algorithm enables efficient gradient-based optimization via backpropagation through the solver's iterative process. The resulting dynamic loss captures the alignment between the residual vector and the Krylov subspace at each iteration, providing a geometrically meaningful proxy for the convergence rate.

The performance of this solver-aware training strategy is validated through numerical experiments, which demonstrate a significant acceleration of iterative solver convergence. The method generalizes across different topological configurations of one-dimensional graphs and adapts to variations in problem discretization without requiring retraining.

These results confirm the effectiveness of the proposed training strategy in enabling the neural preconditioner to directly influence Krylov subspace geometry and, consequently, to accelerate the convergence of iterative solvers.

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Efficient numerical treatment of multidimensional reaction-diffusion PDEs

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Reaction-diffusion partial differential equations (PDEs) play a central role in the mathematical modeling of complex processes arising in sustainability and energy-related applications, including electrochemical phenomena in batteries, vegetation dynamics, and combustion processes. The reliable numerical simulation of these phenomena requires high-order, stable, and efficient time integration methods, particularly in applications involving repeated large-scale PDE solves within iterative optimization procedures for parameter calibration. Among the available approaches, linearly implicit W methods represent a promising class of integrators [5].

In this talk, we therefore focus on two complimentary goals. First, we investigate strategies to improve the efficiency of the above mentioned W methods. In particular, we combine W-methods of order up to four [1,4,6] with Approximate Matrix Factorization (AMF) and matrix-oriented techniques, exploiting the structure of the matrices generated by the discretization of the diffusion operator by means of finite differences. A detailed comparison of the resulting approaches is carried out in terms of computational cost and accuracy [2].

Secondly, we focus on a more general framework of PDEs with time-dependent boundary conditions. In this setting, W methods may experience the phenomenon of order reduction [4]. To address this issue, we propose new boundary correction techniques aimed at restoring the expected convergence order, with a moderate additional computational effort [3]. The effectiveness of the proposed techniques is assessed on several problems with prescribed Neumann, Robin, or mixed Dirichlet-Neumann boundary conditions in multiple space dimensions.

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Hierarchical Clustering and Model Reduction for Optimal Control of ABMs

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Agent-based models (ABMs) describe systems of interacting agents capable of displaying complex collective behaviours such as consensus formation and clustering, with applications in biology and social dynamics [1,2]. However, as the number of agents and the dimension of their states increase, the design of optimal control strategies becomes computationally prohibitive. In this talk, we present a scalable control framework that combines two complementary model reduction strategies: hierarchical clustering, which reduces the number of agents by capturing collective dynamics at the level of cluster centers, and projection-based [3] reduction via Proper Orthogonal Decomposition (POD). These components are integrated within a feedback loop to design optimal control laws on a reduced-order representation of the system. Numerical results on opinion dynamics models [4] show that the proposed approach significantly improves computational efficiency and enables control in regimes where direct methods become infeasible.

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A Feedback-Control Perspective on Sampling for Langevin and McKean–Vlasov Dynamics

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This talk discusses a feedback-control construction for the N -particle interacting system

$$dX_t^i = -\nabla V(X_t^i) dt - \frac{1}{N} \sum_{j=1}^N \nabla W(X_t^i - X_t^j) dt + \sqrt{2\sigma} dB_t^i, \quad i = 1, \dots, N,$$

where $(B^i)_{i=1}^N$ are independent standard Brownian motions and N is large. The feedback is designed from the mean-field PDE, viewed as the Wasserstein gradient flow of the entropy-regularized free energy

$$\mathcal{F}(\mu) := \mathcal{E}(\mu) + \sigma \int_{\mathbb{T}^d} \mu \log \mu \, dx.$$

In the case $W \equiv 0$, this is the Fokker–Planck equation associated with the overdamped Langevin dynamics used for sampling from a Gibbs density. The goal is to use feedback as a controlled burn-in mechanism by accelerating convergence of the empirical measure μ_t^N to a selected equilibrium. This is more needed when $\mathcal{F}$ is nonconvex around $\bar{\mu}$, since the small eigenvalues of the Wasserstein Hessian of $\mathcal{F}$ control the convergence rate.

The feedback is built from the eigenfunctions associated with eigenvalues $\lambda_j \leq \delta$ of the Wasserstein Hessian of $\mathcal{F}$ through an algebraic Riccati equation, and it adds to each particle a drift depending only on the empirical averages of those eigenfunctions. For the closed-loop particle system, conditioned consistency on short blocks, positive-time smoothing of the McKean–Vlasov flow, and Gaussian concentration of the particle Itô map give the estimate

$$\sup_{t_{\text{burn}} \leq t \leq e^{c'} N} \mathbb{E} W_2^2(\mu_t^N, \bar{\mu}) \leq C \beta_d(N) + C e^{-(c-c')N}.$$

Here $\beta_d(N)$ is the empirical-measure rate of [3]. Thus the feedback replaces the usual logarithmic-time obstruction by an exponentially long confinement estimate, within the local basin where the PDE stabilization theorem applies.

For sampling, the normalizing constant of the Gibbs density is usually not available. This work shows how the same controller can be assembled from the unnormalized weight e^{-V} , using the fact that the eigenproblem and the selected eigenfunctions are unchanged. Numerical experiments illustrate this implementation. I will also discuss methods for recovering the selected Hessian eigenfunctions from data.

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C^0 -Conforming Neural Approximated Virtual Element Method on Polygonal Meshes

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We propose two neural-network-based approaches for the construction of C^0 -conforming approximation spaces on general polygonal meshes for the numerical solution of partial differential equations. The proposed techniques are inspired by the Virtual Element Method (VEM) [1], but replace the implicit formulation of local basis functions with explicit neural-network representations to avoid the usage of problem-dependent projection and stabilization operators typically required in VEM. These neural approximations are subsequently employed within a conventional finite element assembly framework.

The original Neural Approximated Virtual Element Method (NAVEM) approximates the VEM basis functions on each element by training neural networks to identify a suitable combination of harmonic functions that mimics the target VEM basis function on the element boundary [2]. However, since the reconstructed basis functions are computed independently on each element, the resulting approximation space is, in general, nonconforming across inter-element boundaries.

To recover C^0 -conformity, we introduce two alternative neural formulations, named B-NAVEM and P-NAVEM, both based on fully connected feed-forward architectures but characterized by different training strategies [3]. In B-NAVEM, the local basis functions are obtained through a Physics-Informed Neural Network (PINN) strategy with strongly enforced Dirichlet boundary conditions. The neural networks are trained to solve the local Laplace problems associated with the VEM basis functions. The resulting basis functions match polynomial VEM traces on the element boundaries and remain approximately harmonic inside the elements, thereby reproducing the main features of the VEM space while ensuring continuity across the mesh. The second approach, P-NAVEM, follows a different philosophy. Rather than reproducing the VEM space itself, it directly learns continuous approximation functions by imposing a polynomial reproducibility condition, which is the essential ingredient for achieving optimal polynomial convergence properties.

Several numerical tests on linear and nonlinear problems confirm that both approaches achieve optimal convergence rates, within the approximation capability of the neural networks, and deliver results comparable to those of the standard VEM, while maintaining the flexibility of polygonal discretizations.

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