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Stabilization Strategies for High Order Methods for Transport Dominated Problems

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Abstract. – *Standard high order Galerkin methods, such as pure spectral or high order finite element methods, have insufficient stability properties when applied to transport dominated problems. In this paper we review some stabilization strategies for pure spectral methods and spectral multidomain approaches.*

1. – Introduction.

1.1 – *General overview.*

Finite element upwind approximation techniques have been developed enormously since the pioneering work by Mitchell and Griffiths [30] and the generalization and analysis by Baba and Tabata [1]. The strongly consistent SUPG–method introduced in 1982 by Brooks and Hughes [8] and analyzed by Johnson, Nävert and Pitkäranta [24] opened the door to high order upwind approximations in a finite element framework. Since then, a broad variety of strategies for determining stabilization parameters, generalizations and other approaches have been proposed, see e.g. the book by Roos, Stynes and Tobiska [35].

1.2 – *Model problem.*

We will concentrate on the following model problem: find $u : \Omega \to \mathbb{R}$ such that

$$\begin{aligned} \mathcal{L}u \equiv -\upsilon \Delta u + \beta \cdot \nabla u + \sigma u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1}$$

where Ω is an open bounded subset of $\mathbb{R}^d$ with boundary $\partial\Omega$. For the problem with non-homogenous boundary conditions standard lifting techniques can be

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based on the BGK model. In order to compute the Couette flow rate $Q_c(\delta, a_1, a_2)$ one can solve numerically Eq. (8.6), extending a finite difference technique first introduced by Cercignani and Daneri [19].

Once $\zeta(H)$ has been numerically evaluated on a grid that spans the domain of interest, Eqs. (9.5) and (9.7) give the pressure field in the gas film as a function of X. Furthermore, a prediction of the vertical force acting on the upper surface of the slider bearing, crucial for practical design, may be obtained from the load carrying capacity W, defined as

(9.9) $$W = \frac{l}{L} \int_0 L/l(P-1)\, dX$$

In order to investigate the effects of the rarefaction parameter δ_o and the bearing number Λ on the basic lubrication characteristics (pressure distribution and load carrying capacity), the parameters describing the gas film geometric configuration were fixed at the following values: $D_1/D_o = 2, L/D_o = 100$. Figure 2 shows the pressure field as a function of the longitudinal coordinate X at three different bearing numbers: $\Lambda = 10,\ 50,\ 200$. To assess the influence of the boundary conditions, the profiles corresponding to different accommodation coefficients (for bounding walls supposed physically identical) are drawn in Figs. 2 and 3.

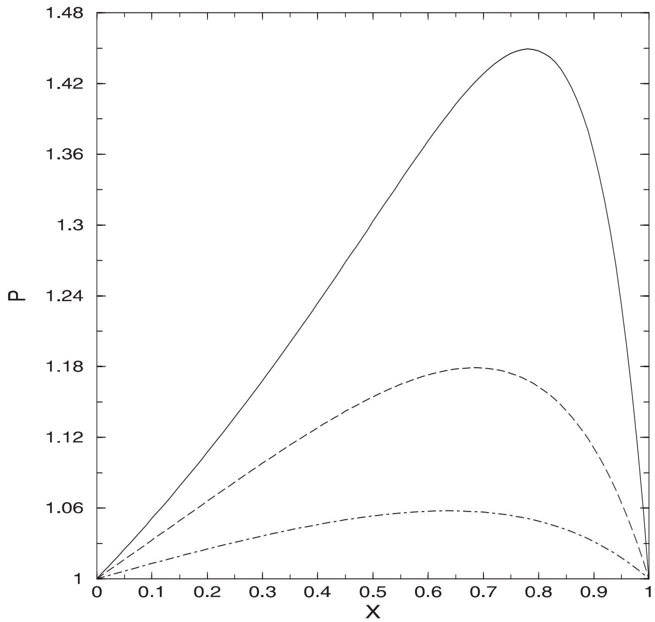

Fig. 3 – Pressure profile for $\delta_o = 0.5$. The line styles indicate $a = 0.8$ (solid), $a = 0.3$ (dashed), and $a = 0.1$ (dot dashed). The bearing number Λ is 200.

surable functions $x \longrightarrow u(x) : (0,1) \longrightarrow H$, whose p^{th} power norms are summable, with the norm $\|u\|_{0,p}^p = \int_0^1 \|u(x)\|_H^p dx, p \in (1,\infty)$.

The vector-valued Sobolev space is defined as

$$W_p^n(0,1; D(A^2), H) = \{u : A^2 u \in L^p(0,1;H);\ u^{(n)} \in L^p(0,1;H)\},$$

the norm in this space is given by

$$\|u\|_{W_p^n(0,1;D(A^2),H)} = \left\|A^2 u\right\|_{L^p(0,1;H)} + \left\|u^{(n)}\right\|_{L^p(0,1;H)}.$$

The space $D(A)$ is defined by

$$D(A) = \left\{u \in D_A; \|u\|_{D(A)}^2 = \|u\|_H^2 + \|Au\|_H^2 < \infty\right\}$$

and it is precisely the domain of A equipped with the Hilbertian graph norm.

Let $-A$ be the generator of the analytic semigroup e^{-tA} for $t > 0$, decreasing at infinity, and strongly continuous for $t \geq 0$. We define the interpolation space [14, p. 96]:

$$(H, D(A^m))_{\theta,p} = \left\{u \in H; \|u\|_{m,\theta}^p = \int_0^\infty t^{m(1-\theta)p-1} \left\|A^m e^{-tA} u\right\|_H^p dt < \infty\right\},$$
$$0 < \theta < 1, m \in \mathbb{N}, 1 < p < \infty.$$

$\|.\|_{m,\theta}$ is the norm in $(H, D(A^m))_{\theta,p}$.

Let $(Fu)(\sigma) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{+\infty} e^{-i\sigma x} u(x) dx$ be the Fourier transform of u.We recall the following.

DEFINITION 1. – *The mapping $T : \mathbb{R} \longrightarrow B(H)$, is said to be a Fourier multiplier of the type (p,p) if for all $f \in L^p(\mathbb{R}, H)$ we have*
$\left\|F^{-1}TFf\right\|_{L^p(\mathbb{R},H)} \leq C\|f\|_{L^p(\mathbb{R},H)}$.

THEOREM 2 (Mikhlin-Schwartz) [7]. – *If the mapping*

$$\begin{array}{cccc} T: & \mathbb{R} & \mapsto & B(H) \\ & \sigma & \mapsto & T(\sigma) \end{array}$$

is continuously differentiable, and the inequalities

$$\|T(\sigma)\|_{B(H)} \leq C \text{ and } \left\|\frac{\partial}{\partial\sigma} T(\sigma)\right\|_{B(H)} \leq \frac{C}{|\sigma|} \text{ for } \sigma \neq 0$$

hold for all $\sigma \in \mathbb{R}$, then $T(\sigma)$is a Fourier multiplier of type (p,p).

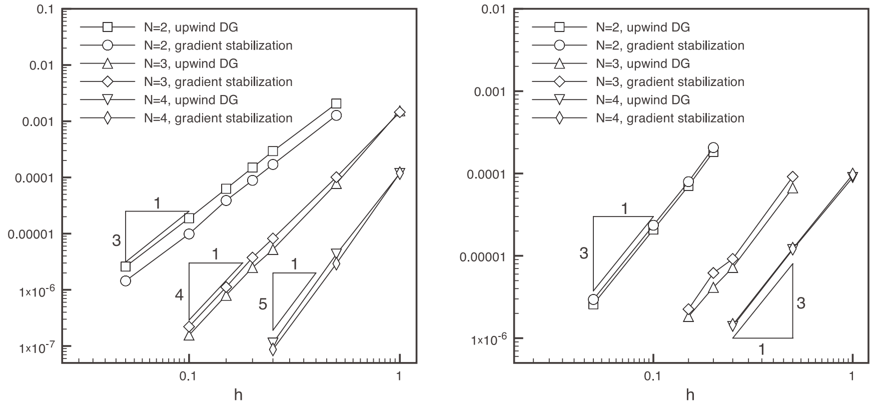

Fig. 4. – Convergence behavior of the upwind DG-method and the DG-method using the jump of the gradient for the advection-reaction equation under h-refinement for a smooth solution (left) and a irregular solution (right) measured in the L^2-norm.

6. – Conclusion.

Comparing the DG- with the continuous versions of the finite element method is not easy. A discontinuous finite element space allows easier implementation of *hp*-strategies with hanging nodes. On the other hand the disadvantage is the increased number of degrees of freedom. But this argument gets less and less relevant while increasing the local polynomial degree N since basis-functions having its support on the edges increase proportionally to N while basis-functions having their support only on the interior of the element increase as N^2. In general we conclude that DG-methods are successful for pure transport problems and stabilization parameter independent local mass conservation can be achieved, see (13). For convection-diffusion problems continuous approximations are attractive since the stabilization can be made independent of the diffusion parameter. A comparison with respect to different aspects of the continuous methods can be found in Table 1.

TABLE 1. – Comparison of some relevant aspects for different schemes.

	SUPG	OS	SV	LPS	CIP
Time-stepping, source terms	–	+	+	+	+
Stabilization par. indep. on diffusion coefficient	–	+	+	+	+
Same degree of freedom as standard FEM	+	–	+	+	+
Same stencil as standard FEM	+	+	–	–	–
Data-structure (*)	+	+	–	–	–

(*) Need use of hierarchical meshes or to know solution on heighbor elements, in contrast to standard FEM.

Model sheet of the Bollettino

REFERENCES

[1] S. AGMON - L. NIRENBERG, *Properties of Solutions of Ordinary Differential Equation in Banach Spaces*, Comm. Pure Appl. Math., **16** (1963), 121-239.

[2] M. S. AGRANOVICH - M. L. VISIK, *Elliptic Problems with a Parameter and Parabolic Problems of General Type*, Russian Math. Surveys, **19** (1964), 53-161.

[3] A. AIBECHE, *Coerciveness Estimates for a Class of Elliptic Problems*, Diff. Equ. Dynam. Syst., **4** (1993), 341-351.

[4] A. AIBECHE, *Completeness of Generalized Eigenvectors for a Class of Elliptic Problems*, Result. Math., **31** (1998), 1-8.

[5] F. COBOS - D. L. FERNANDEZ, *On Interpolation of Compact Operators*, Ark. Mat., **27** (1989), 211-217.

[6] M. DENCHE, *Abstract Differential Equation with a Spectral Parameter in the Boundary Conditions*, Result. Math., **35** (1999), 217-227.

[7] N. DUNFORD - J. T. SCHWARTZ, *Linear Operators*, Vol. **II**, Interscience, New York (1963).

[8] A. FAVINI - J. A. GOLDSTEIN - S. ROMANELLI, *An analytic semigroup associated to a degenerate evolution equation*, in *Stochastic Processes and Functional Analysis*, J. A. Goldstein, N. E. Gretsky and J. J. Uhl Jr. eds, Marcel Dekker, New York, (1997), 88-100.

[9] A. FAVINI - A. YAGI, *Degenerate Differential Equations in Banach Spaces*, Marcel Dekker, New York (1999).

[10] S. G. KREIN, *Linear Differential Equations in Banach Spaces*, American Mathematical Society, Providence (1971).

[11] J. L. LIONS, *Sur les Espaces d'Interpolation: Dualité*, Math. Scand., **9** (1961), 147-177.

[12] J. L. LIONS - E. MAGENES, *Problèmes aux Limites non Homogènes et applications*, Vol. **I**, Dunod, Paris (1968).

[13] J. L. LIONS - J. PEETRE, *Sur une classe d'espaces d'interpolation*, Inst. Hautes Etudes Sci. Publ. Math., **19** (1964), 5-68.

[14] H. TRIEBEL, *Interpolation Theory, Functions Spaces, Differential Operators*, North Holland, Amsterdam (1978).

[15] S. Y. YAKUBOV, *Completeness of Root Functions of Regular Differential Operators*, Longman, Scientific and Technical, New York (1994).

[16] S. Y. YAKUBOV, *Linear Differential Equations and Applications*, Baku, elm (1985). (in Russian).

[17] S. Y. YAKUBOV, *Noncoercive Boundary Value Problems for the Laplace Equation with a Spectral Parameter*, Semigroup Forum, **53**, (1996), 298-316.

[18] S. Y. YAKUBOV - Y. Y. YAKUBOV, *Differential-Operator Equations. Ordinary and Partial Differential Equations*, Chapman and Hall/CRC Press, New York (2000).

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